Assimilation de Données, Observabilité et Quantification d'Incertitudes appliqués à l'hydraulique à surface libre

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Colloque National d'Assimilation
2 décembre 2016

Plan

- Systèmes étudiés : canaux d'irrigation, fleuves
- Questions : problèmes inverses & incertitudes
- Modèles :
    - Saint-Venant 1D-1.5D non linéaire (SIC)
    - Modèle Linéaire Tangent, Adjoint
- Filtre de Kalman, pour un canal d'irrigation (X, Q_p, C_d)
- FK, Evolution des Erreurs, Observabilité
- 4D-Var, données SWOT sur la Garonne
- 4D-Var, quantification d'incertitudes
- Conclusions

Irrigation canal

- Water for irrigation, industries and domestic uses
- Automatic gates
- SCADA systems

Control objectives

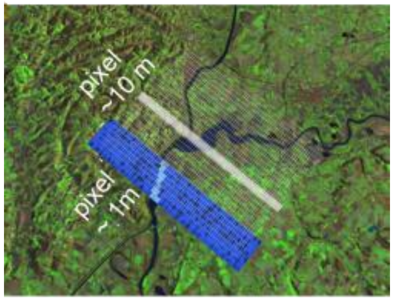
- Distribute water to users (offtakes)
- Minimize water losses
- Satisfy constraints (levels, discharges, etc)

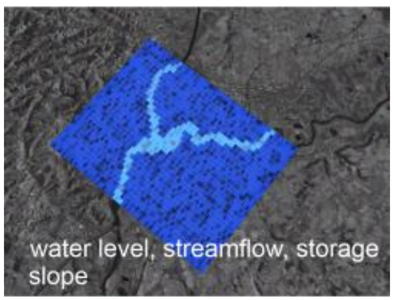
- By mean of control actions (manual or automatic) on cross devices (gates, weirs, pumps)
- Automatic control = type of inverse problem

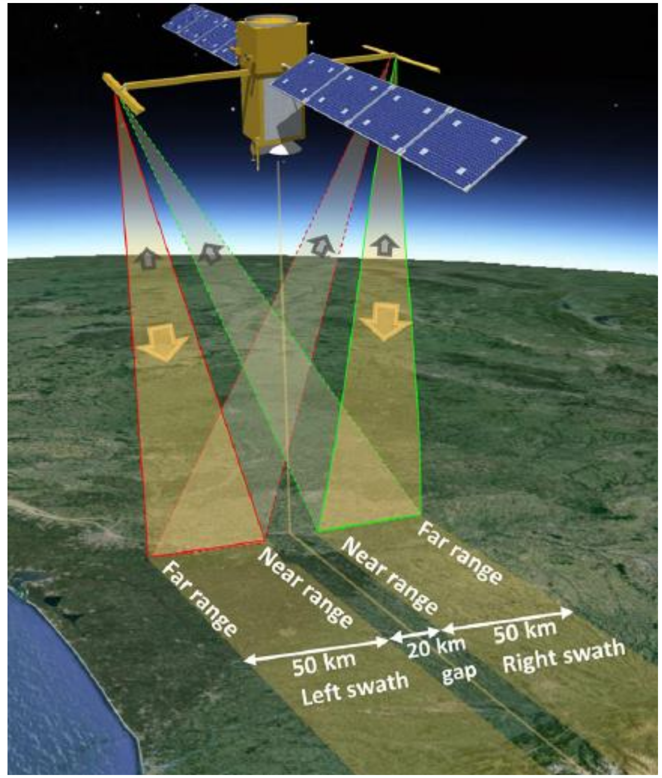
Inverse problems for controlled Irrigation Canals

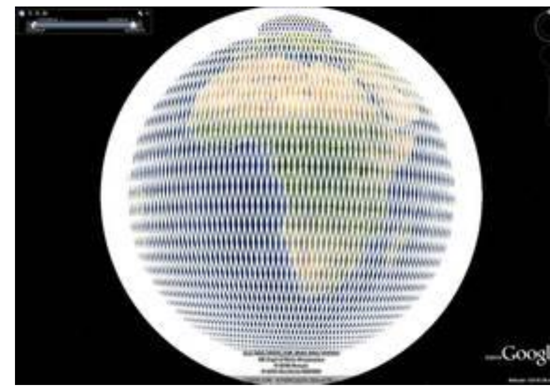
- Known:
  - Bathymetry, boundary conditions (up, mid, down)
- Observations:
  - Water levels close to cross devices
- Unknowns / Uncertainties (active & passive):
  - Offtake withdrawal or inflow (Q_p)
  - Cross device discharge coefficients (C_d)
  - Hydraulic state (Q, Z) (e.g. state feedback ctrl)
  - Friction coefficients
- Objective: water loss or theft, fault detection, model update for operation & maintenance, etc.

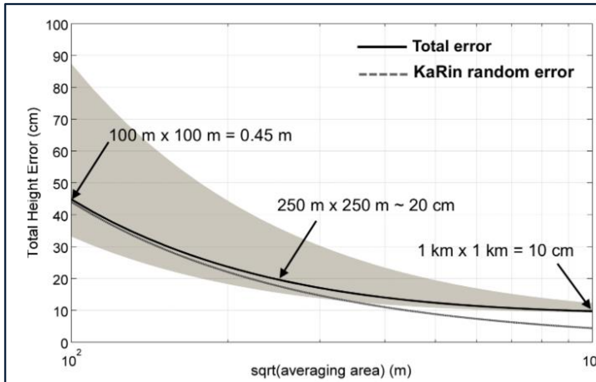
Surface Water and Ocean Topography (SWOT) mission (2021)











Scientific requirements

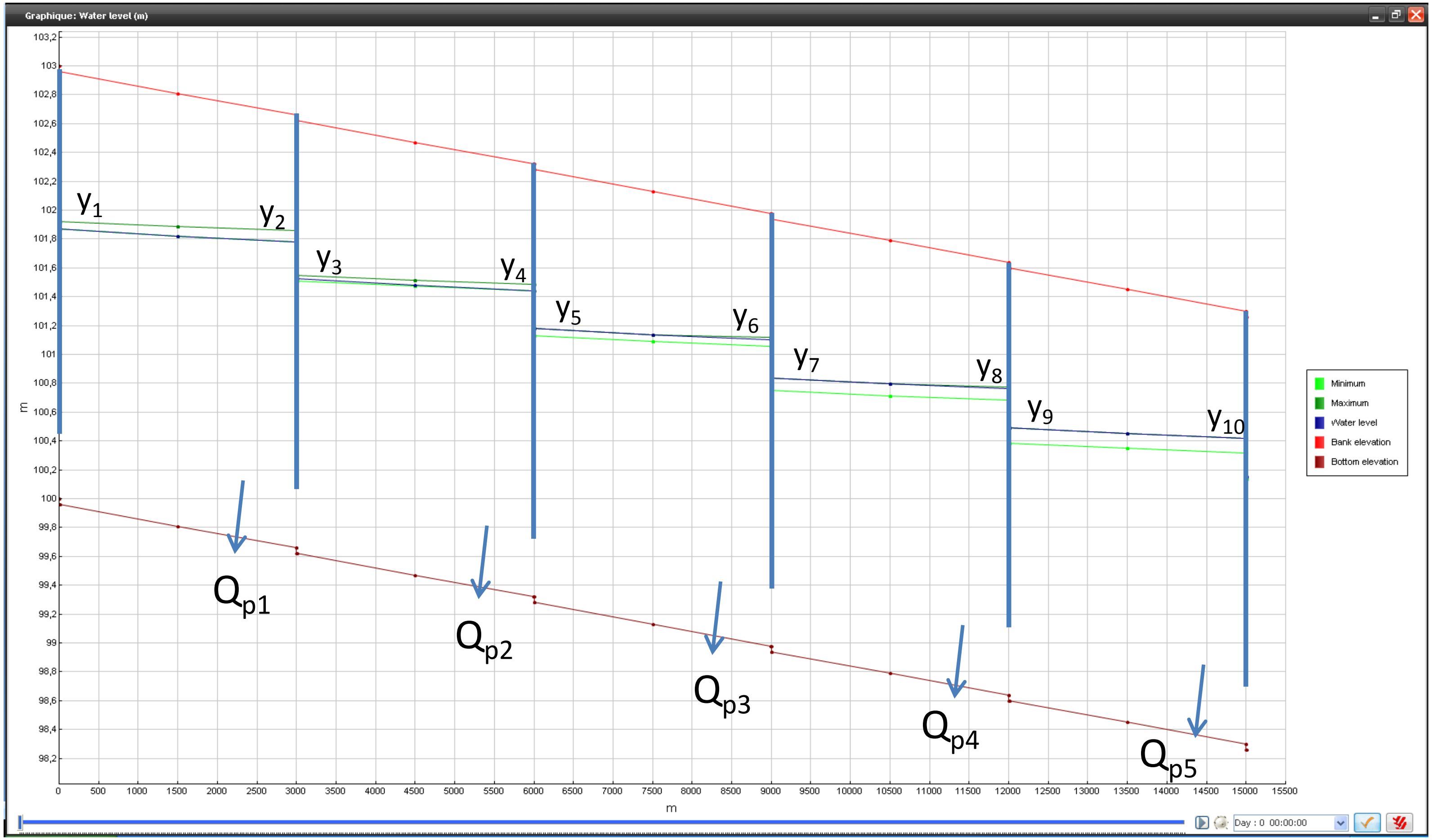
Observable river width	> 100 m
Height accuracy	10 cm over area > 1 km²
Slope accuracy	1.7 cm/km over area > 1 km
Width accuracy	15% of the evaluated river
Data collection	90% of all ocean/continents within the orbit during 90% of the operational time.

http://www.aviso.altimetry.fr/en

Inverse problems for rivers observed from space

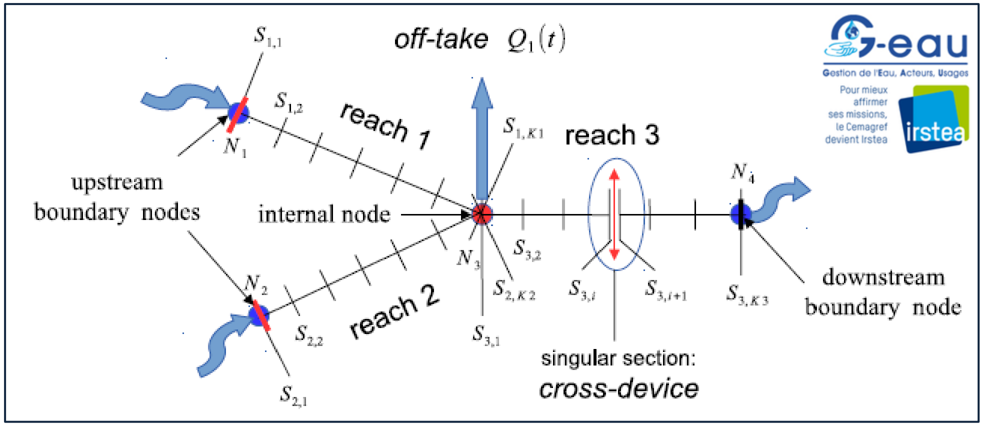
- Known:
  - River length, mean slope, mean annual discharge !?
  - boundary condition (down) !?
- Observations:
  - Water levels locally, or globally (SWOT swaths), local slope, width
- Unknowns / Uncertainties (active & passive):
  - <u>Boundary conditions</u> (Q_{up})
  - <u>Bathymetry, Friction</u> (strong effect)
  - Hydraulic state (Q, Z) outside observations windows (time and space)
  - Tributaries inflows (Q_p)
- Objective: flood control, resource availability, navigation, water balance

Example (5-pool irrigation canal)



5 pools, 5 unknown withdrawals Q_p , 10 water level measurements Y

SIC² : Simulation and Integration for Controls of Canals



- 1D/1.5D hydraulic model
  (Multiple beds + Storage areas)
- Flow dynamics of rivers
- Irrigation canals, drainage network, etc.

- Based on the full Saint-Venant equations
- TLM & Adjoint (Tapenade)

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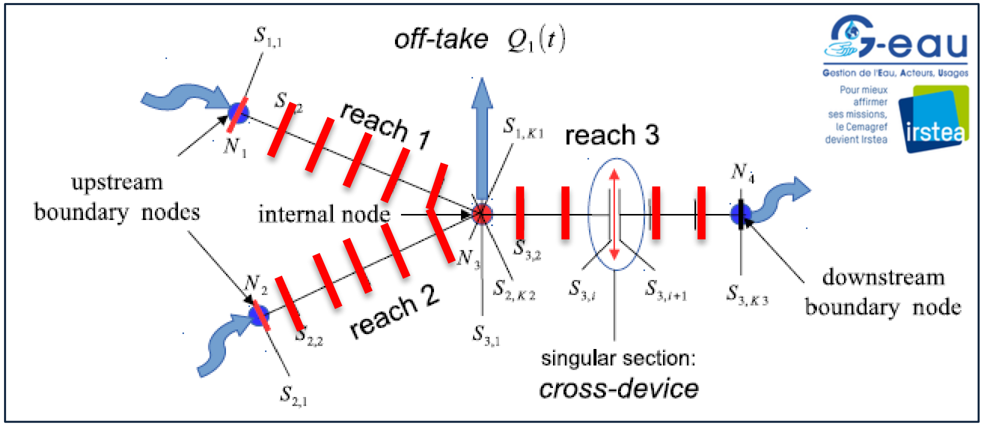

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▪ Regular sections :

$$\partial_t A + \partial_x Q = Q_L$$
$$\partial_t Q + \partial_x (Q^2 / A) + gA\partial_x Z = -gAS_f + C_k Q_L v \,, \qquad S_f = \frac{Q^2}{K_S^2 A^2 R^{4/3}}, \qquad t \in [0, T]$$

- $Q(x,t)$: Discharge
- $Z(x,t)$: Water level
- $A(x,t)$: Wetted cross–sectional area
- $v(x,t) = Q/A$: Mean velocity
- R: Hydraulic radius
- K_S: Strickler coefficient
- C_k: Lateral discharge coefficient

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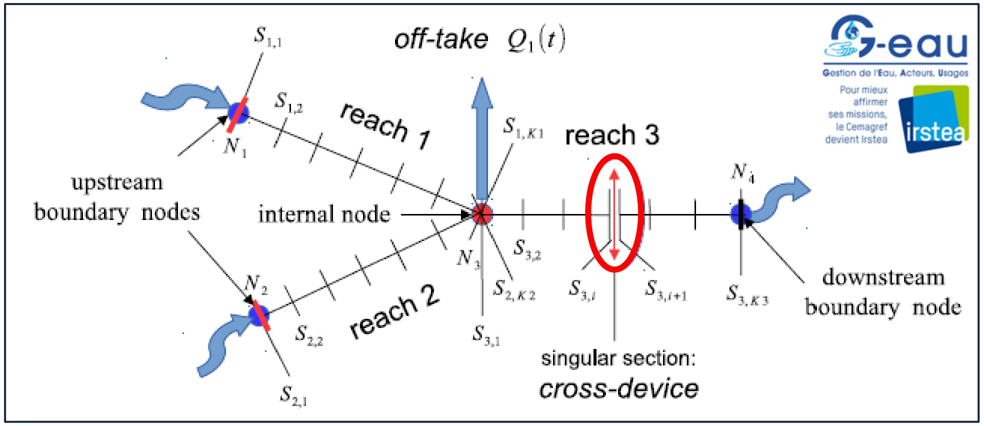

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Singular sections :

$$Q_{|S_{3,i}} - Q_{|S_{3,i+1}} = 0, \quad Q_{|S_{3,i}} = \mathcal{F}(Z_{|S_{3,i}}, Z_{|S_{3,i+1}}, C_{d\,|S_{3,i}})$$

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- $S_{i,k}$: k^{th} cross section of the reach i

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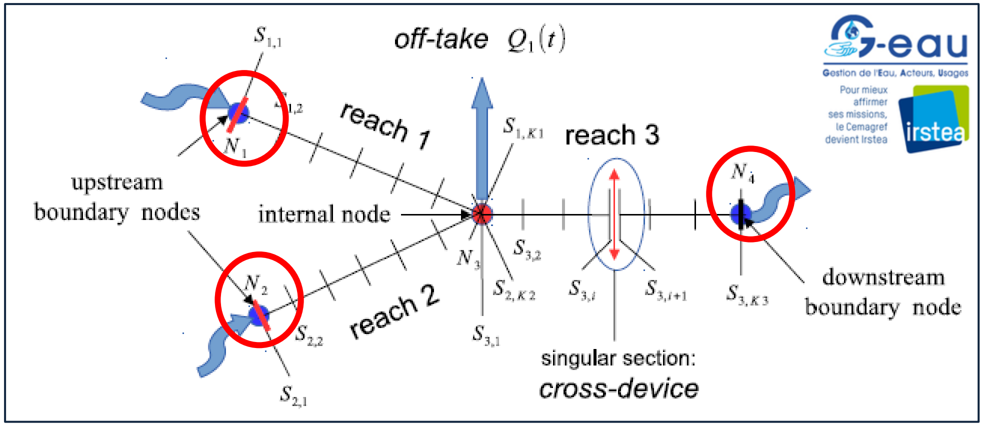

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Boundary conditions:

- *Upstream nodes* $: Q(t) \; or \; Z(t)$
- *Downstream nodes* $: Q(t) = f(Z(t), p_{rc})$

- $Q(x,t)$: Discharge
- $Z(x,t)$: Water level
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- K_S: Strickler coefficient
- C_k: Lateral discharge coefficient
- C_d: Cross device coefficient
- $S_{i,k}$: k^{th} cross section of the reach i
- p_{rc}: rating curve parameters

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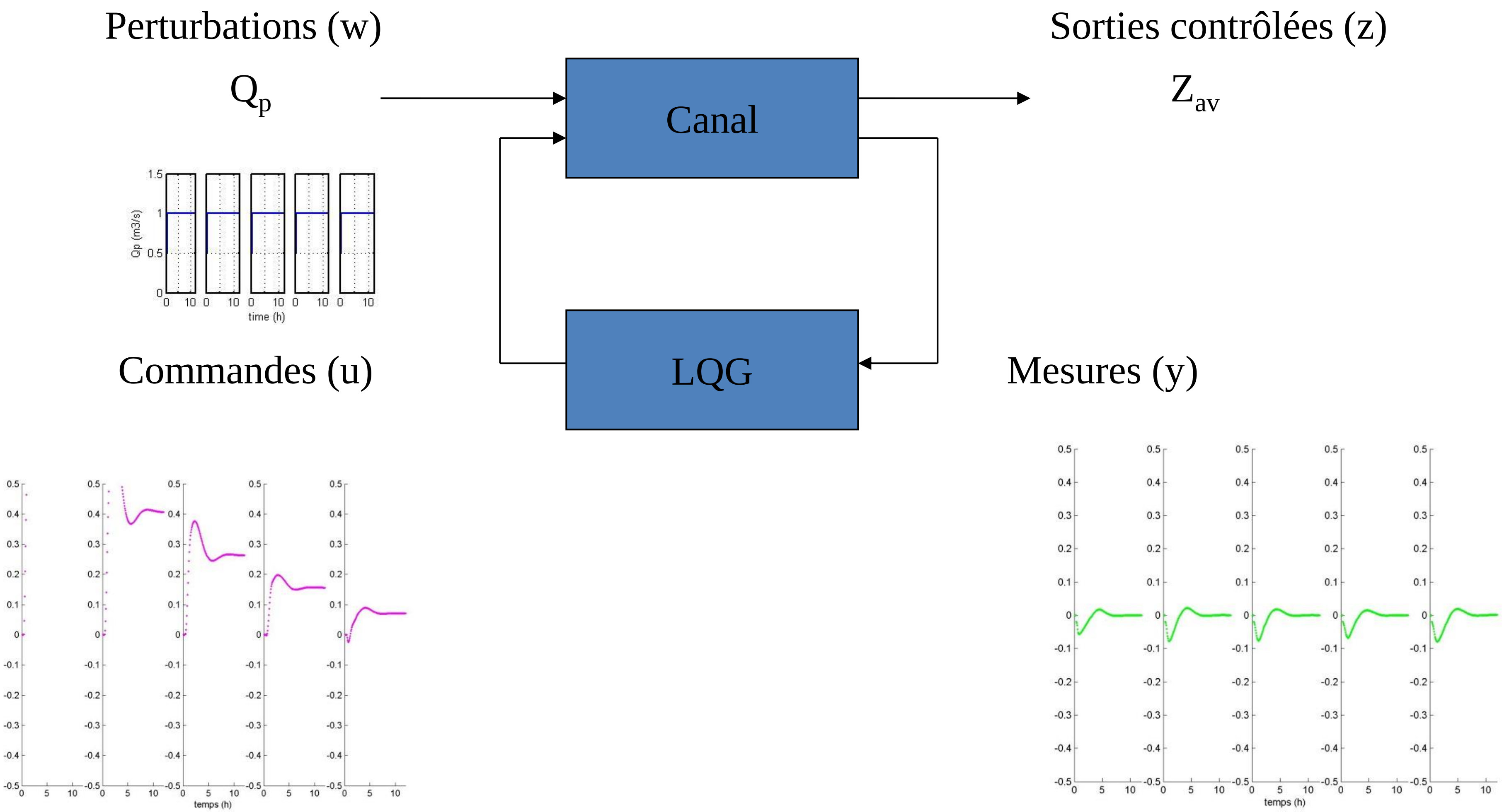
LQG (Linear Quadratic Gaussian)



Design du contrôleur LQG

- Minimisation d'un critère J :

$$J = \frac{1}{2}\sum_{k=0}^{N}\left\{(y(k) - y^*(k))^T . Q_y . (y(k) - y^*(k)) + u(k)^T . R . u(k)\right\}$$

- Contraintes dynamiques :

$$\begin{cases} x(k+1) = A.x(k) + B.u(k) + B_p.u_p(k) + B_c.u_c(k) \\ y(k) = C.x(k) \end{cases}$$

u : vecteur des ouvertures des 5 vannes

u_p : vecteur des débits aux 5 prises

u_c : vecteur des coefficients de débit aux 5 ouvrages en travers

Solution optimale

- Solution optimale (retour d'état) :

$$u(k) = -K_{LQ}(k).x(k)$$

K_{LQ} matrice gain du contrôleur LQ, solution d'une équation de Riccati

Observateur d'état et de perturbation

$$\begin{cases} \hat{x}^+ = A.\hat{x} + B.u + B_p.\hat{u}_p + B_c.\hat{u}_c + L.(y - \hat{y}) \\ \hat{u}_p{}^+ = \hat{u}_p + L_p.(y - \hat{y}) \\ \hat{u}_c{}^+ = \hat{u}_c + L_c.(y - \hat{y}) \\ \hat{y} = C\hat{x} \end{cases}$$

L, L_p, L_c : par placement de pole (A-LC) pour avoir convergence de l'erreur de reconstruction de l'état x vers 0 => **Observateur de Luenberger**
(théorie de l'observabilité, théorie de la détectabilité, principe de séparation)

L, L_p, L_c : Minimisation de l'erreur (variance) de reconstruction => **Filtre de Kalman**

Kalman Filter equations

Initial linear model + noise with given covariance matrices Q and R

$$\begin{cases} \begin{pmatrix} x \\ u_p \\ u_c \end{pmatrix}_{k+1} = \begin{pmatrix} A & B_p & B_c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ u_p \\ u_c \end{pmatrix}_k + \begin{pmatrix} B \\ 0 \\ 0 \end{pmatrix} u_k + w_k \\ y_k = \begin{pmatrix} C & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ u_p \\ u_c \end{pmatrix}_k + v_k \end{cases}$$

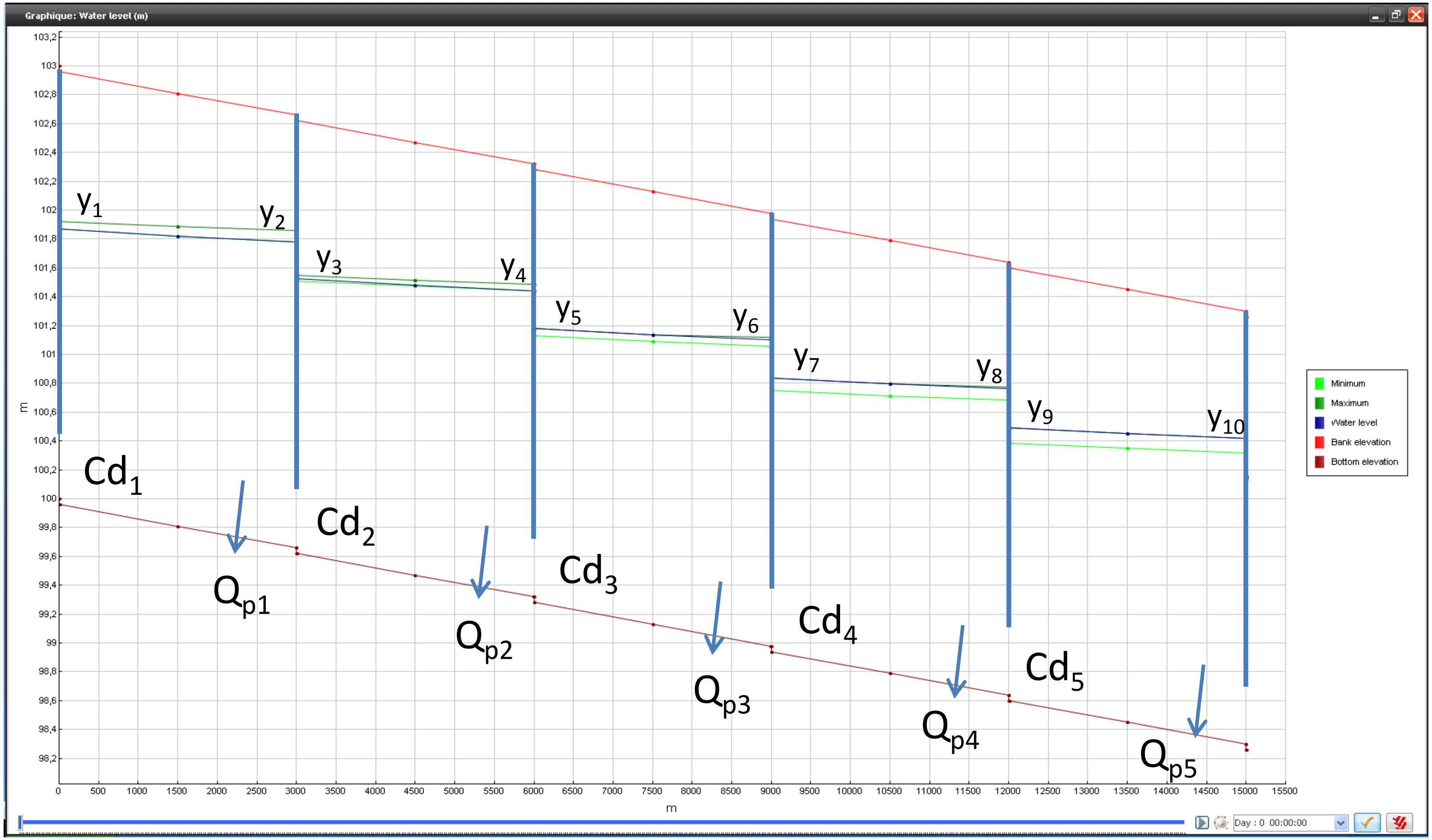
Augmented state for the reconstruction of x, u_p (Q_p) and u_c (C_d)

$$\begin{cases} K_k = P_k^- \tilde{C}^t \left(\tilde{C} P_k^- \tilde{C}^t + R \right)^{-1} \\ \hat{X}_k = \hat{X}_k^- + K_k \left(Y_k - \tilde{C} \hat{X}_k^- \right) \\ P_k = \left(I - K_k \tilde{C} \right) P_k^- \end{cases}$$

$$\begin{cases} \hat{X}_k^- = \tilde{A} \hat{X}_{k-1} + \tilde{B} U_{k-1} \\ P_k^- = \tilde{A} P_{k-1} \tilde{A}^t + Q \end{cases}$$

No observability requirement !!!

Uncertainty reduction (on the 5 Q_p and 5 C_d)



Is it possible to identify Q_p and C_d from limited water level measurements Y ?

Scenario tested with a Kalman Filter

- Time step for the simulations is 300 seconds. T_{end} = 48h
- The 5 offtakes are operated at time 6 h (-0.100 m3/s) and again at time 20 h (+0.100 m3/s) returning to the initial discharge values of -0.175 m3/s
- All cross regulators keep the same gate positions (open loop), but the discharge coefficient of their gates is changed from 0.82 to 0.66 at time 30 mn, then to 0.72 at time 6 h and returning to initial value 0.82 at 24 h
- The initial state X_0 is taken using random variables
- For the matrix Q, we choose $\sigma_Q = 0.1\ m^3/s$ and $\sigma_Z = 0.1$ m for the normal hydraulic states and $\sigma_{QC} = 1$ for the augmented states (Q_p and C_d). For the matrix R, we choose $\sigma_R = 0.01$ m
- There are 10 water level measurements ($y_1, y_2, \dots, y_{10}$), or less !

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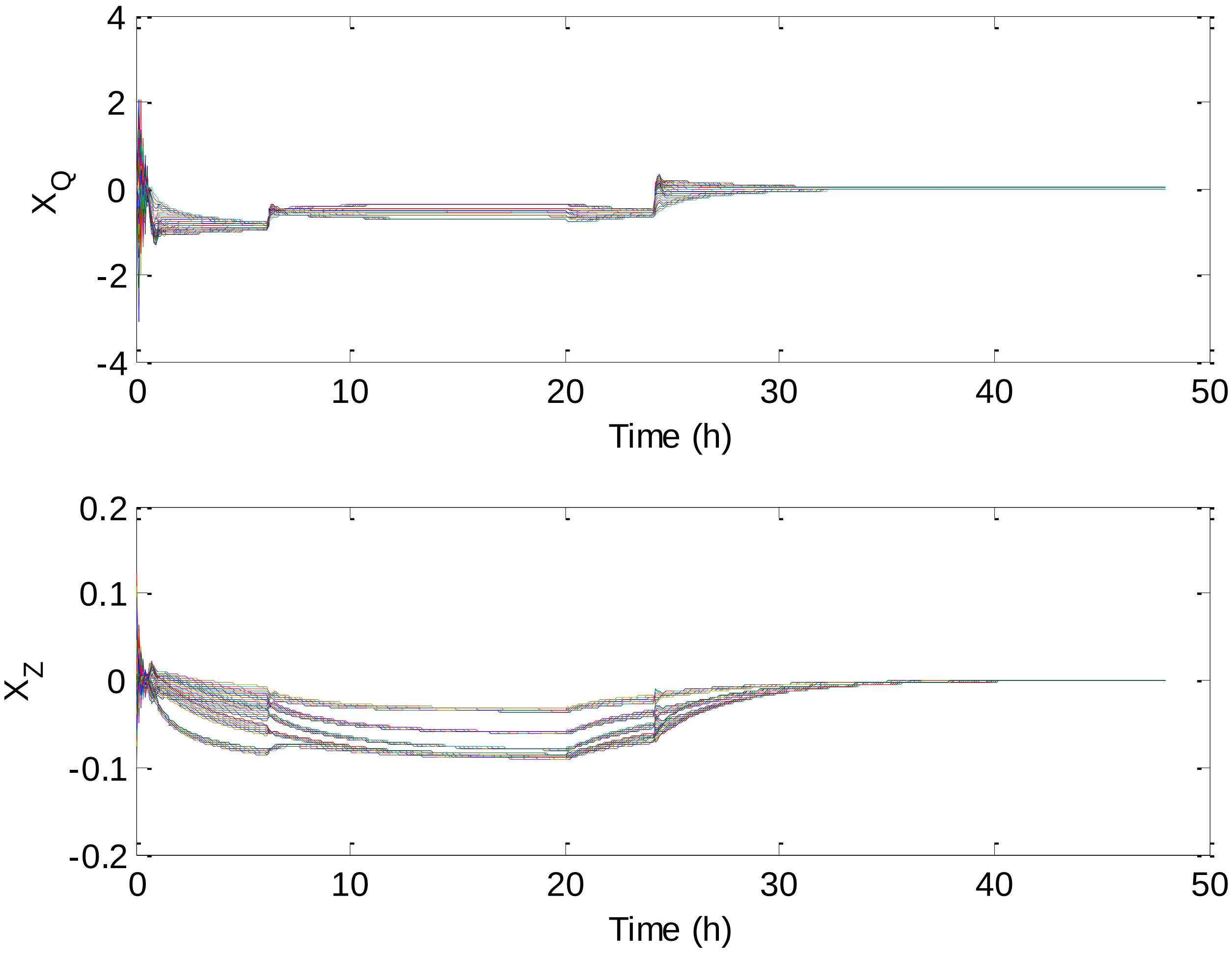
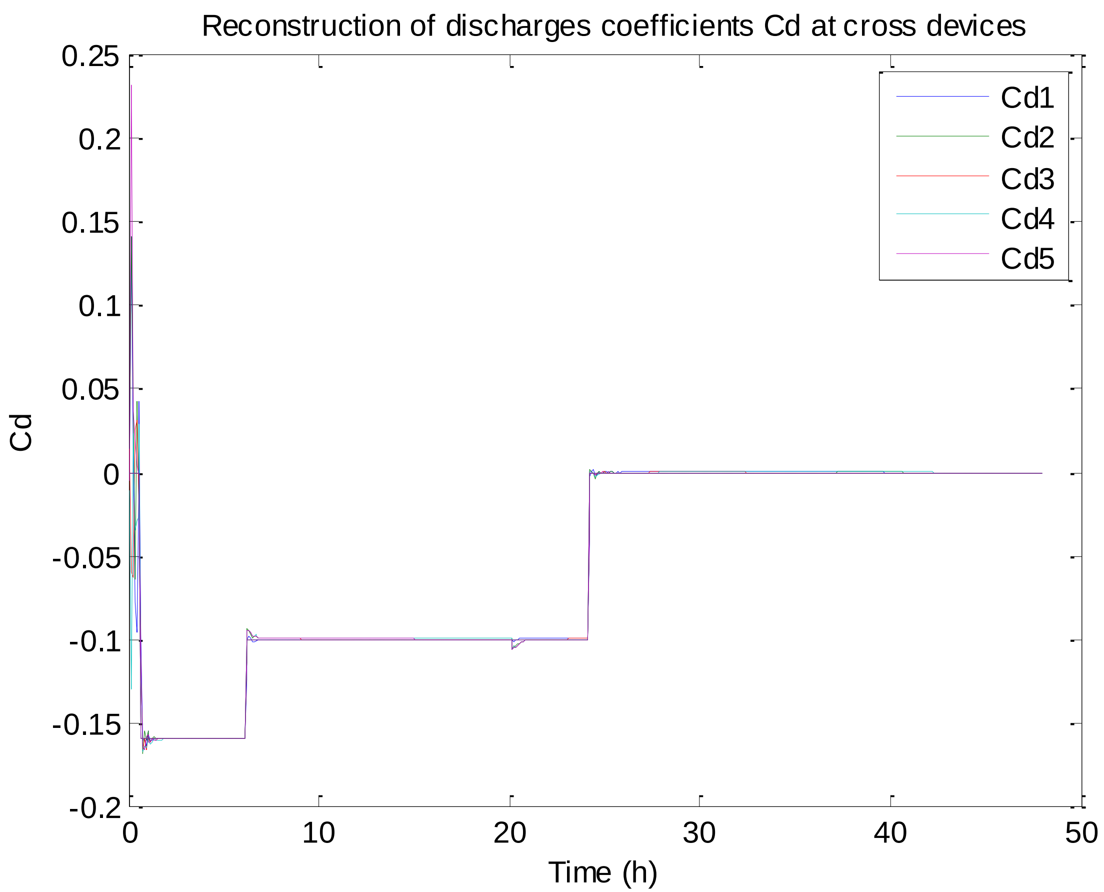
State reconstruction – Twin exp – X (Q & Z)

nx=125

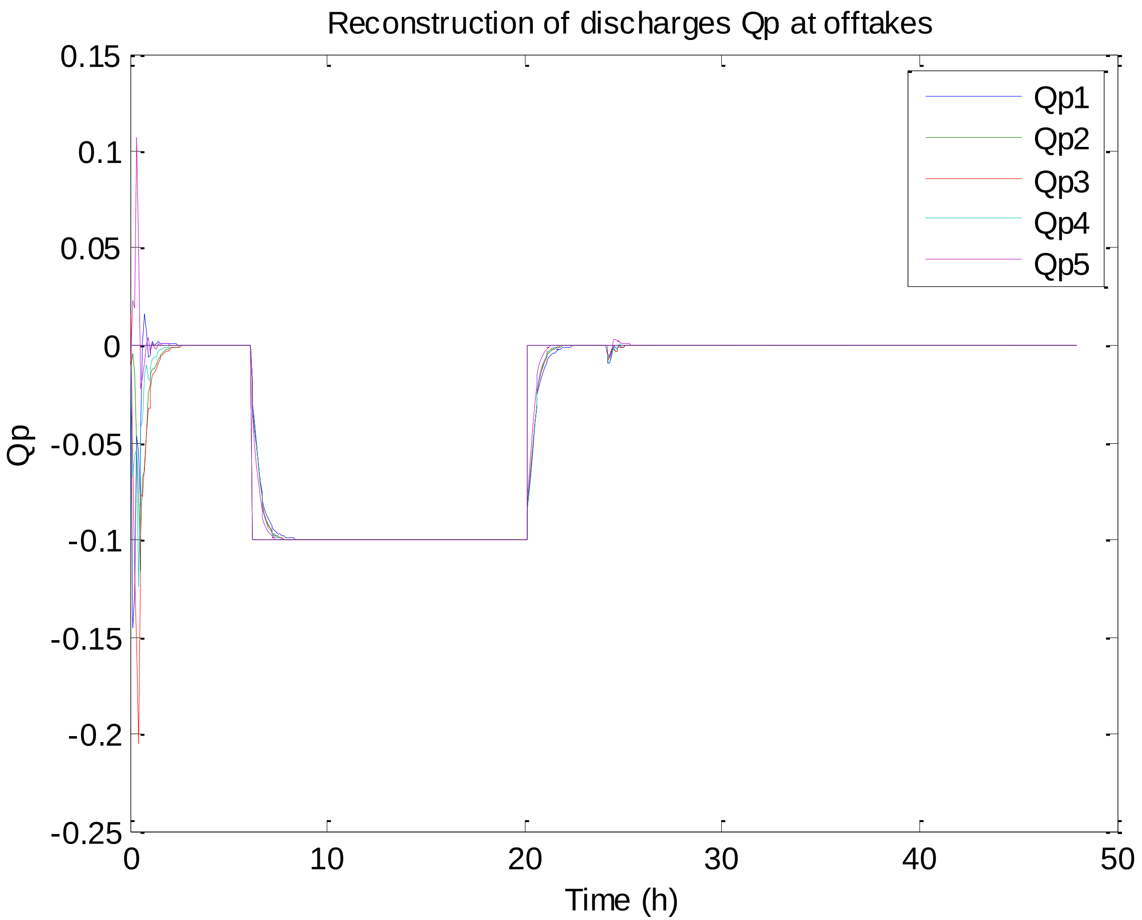


State reconstruction – Twin exp - C_d



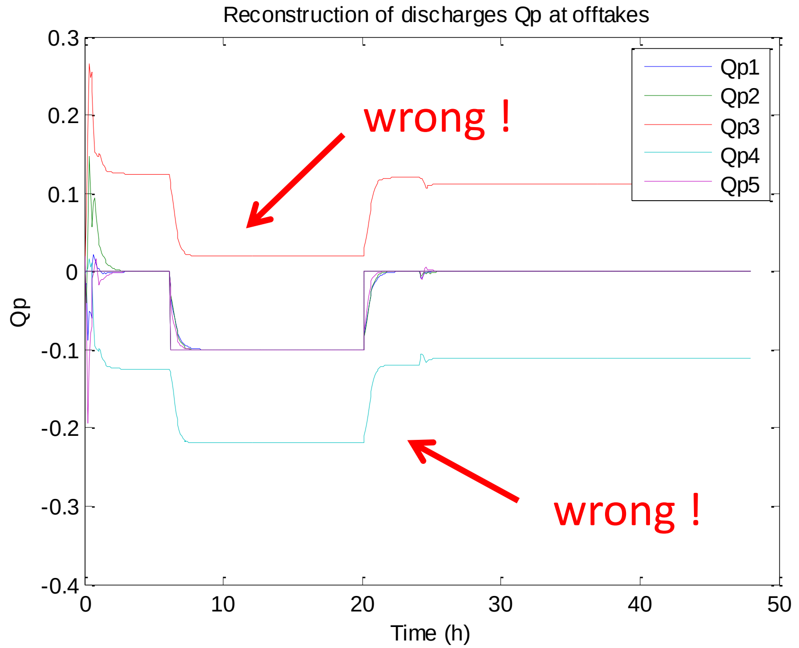
Good and fast convergence
Despite wrong initial state X_0
(No additional noise)

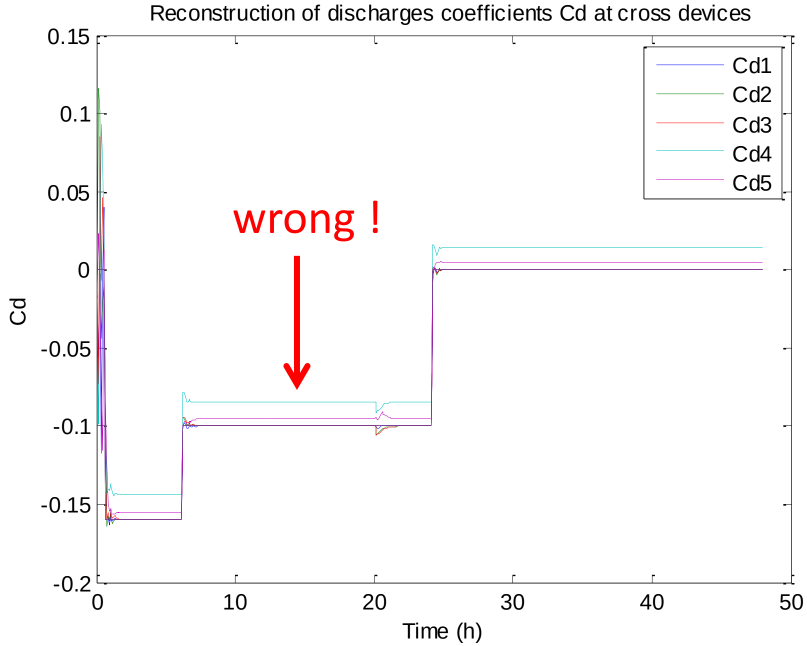
State reconstruction – Twin exp - Q_p



Good and fast convergence
Despite wrong initial state X_0
(No additional noise)

No convergence towards the correct values (case with 9 measurements only: y_8 removed)

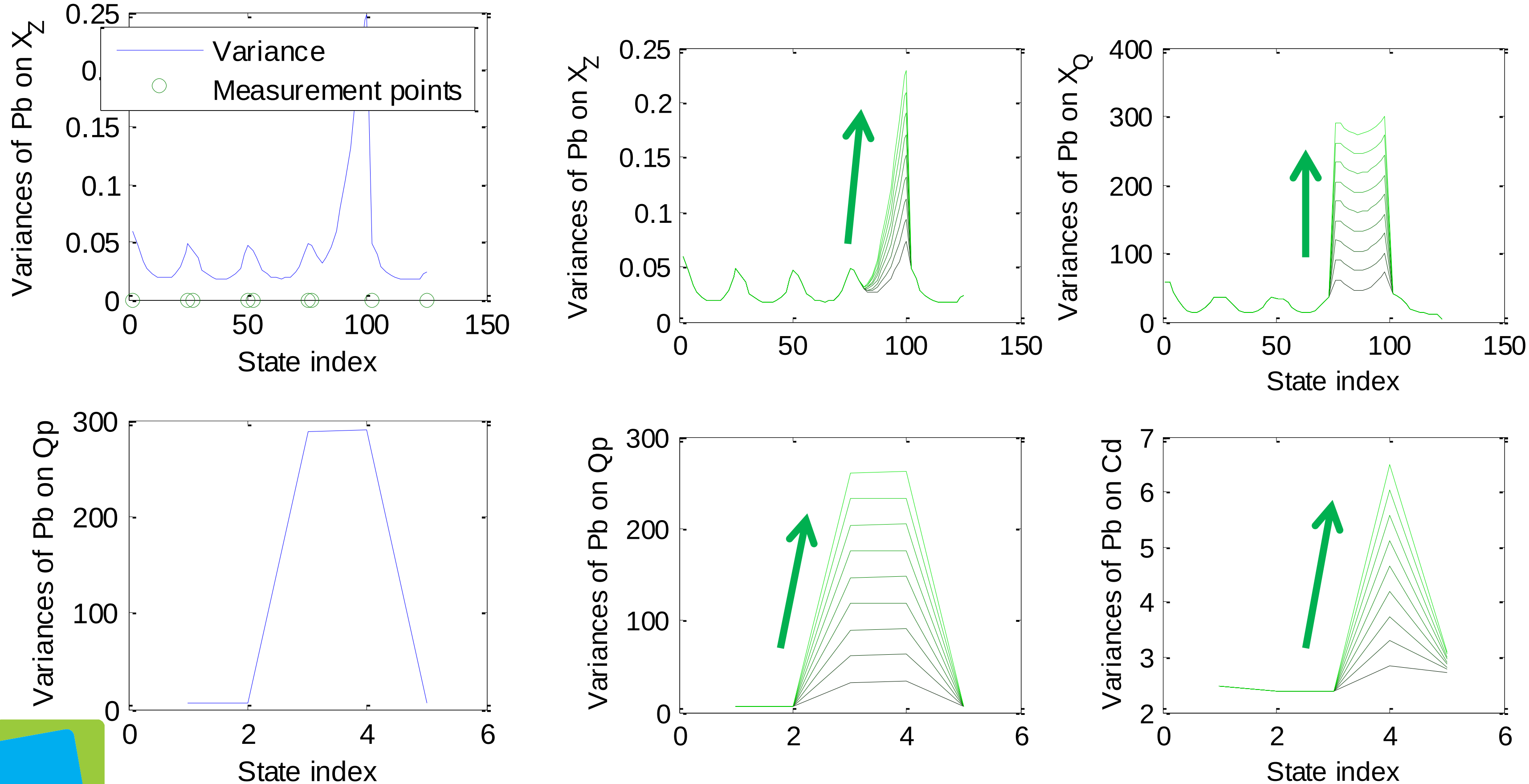




Timing is good but not the values

Very bad for Q_{p3} and Q_{p4} (the 2 influenced a lot by y_8), surprisingly good for C_d (except C_{d4} for the same reasons)

Increase of the local variance (close to y_8) during time iterations: we know that the values are not good



Convergence study

We want that the estimation error converges toward 0

$$\varepsilon_{k+1}^{a} = X_{k+1}^{a} - X_{k+1}^{t}$$

$$\varepsilon_{k+1}^{a} = X_{k+1}^{b} + K_{k+1}\left(Y_{k+1} - C_{k+1}X_{k+1}^{b}\right) - X_{k+1}^{t}$$

$$\varepsilon_{k+1}^{a} = A_k X_k^{a} + B_k U_k + K_{k+1}\left(Y_{k+1} - C_{k+1}X_{k+1}^{b}\right) - A_k X_k^{t} - B_k U_k - \eta_k$$

$$\varepsilon_{k+1}^{a} = A_k X_k^{a} + K_{k+1}\left(C_{k+1}X_{k+1}^{t} + \varepsilon_{k+1}^{o} - C_{k+1}A_k X_k^{a} - C_{k+1}B_k U_k\right) - A_k X_k^{t} - \eta_k$$

$$\varepsilon_{k+1}^{a} = A_k X_k^{a} + K_{k+1}\left(C_{k+1}A_k X_k^{t} + C_{k+1}B_k U_k + C_{k+1}\eta_k + \varepsilon_{k+1}^{o} - C_{k+1}A_k X_k^{a} - C_{k+1}B_k U_k\right) - A_k X_k^{t} - \eta_k$$

$$\varepsilon_{k+1}^{a} = A_k X_k^{a} + K_{k+1}\left(C_{k+1}A_k X_k^{t} + C_{k+1}\eta_k + \varepsilon_{k+1}^{o} - C_{k+1}A_k X_k^{a}\right) - A_k X_k^{t} - \eta_k$$

$$\varepsilon_{k+1}^{a} = A_k\left(X_k^{a} - X_k^{t}\right) - K_{k+1}C_{k+1}A_k\left(X_k^{a} - X_k^{t}\right) + K_{k+1}C_{k+1}\eta_k + K_{k+1}\varepsilon_{k+1}^{o} - \eta_k$$

$$\varepsilon_{k+1}^{a} = \left(A_k - K_{k+1}C_{k+1}A_k\right)\varepsilon_k^{a} + \left(K_{k+1}C_{k+1} - 1\right)\eta_k + K_{k+1}\varepsilon_{k+1}^{o}$$

$$E\left[\varepsilon_{k+1}^{a}\right] = \left(A_k - K_{k+1}C_{k+1}A_k\right)E\left[\varepsilon_k^{a}\right] \text{si } E\left[\eta_k\right] = 0 \text{ et } E\left[\varepsilon_k^{0}\right] = 0$$

$$\Leftrightarrow$$

$$\lim_{k\to\infty} E\left[\varepsilon_{k+1}^{a}\right] = \lim_{k\to\infty}\left(A_k - K_{k+1}C_{k+1}A_k\right)\left(A_{k-1} - K_k C_k A_{k-1}\right)\cdots\left(A_0 - K_1 C_1 A_0\right)E\left[\varepsilon_0^{a}\right] \overset{?}{=} 0$$

Convergence condition :

If for k > k_n

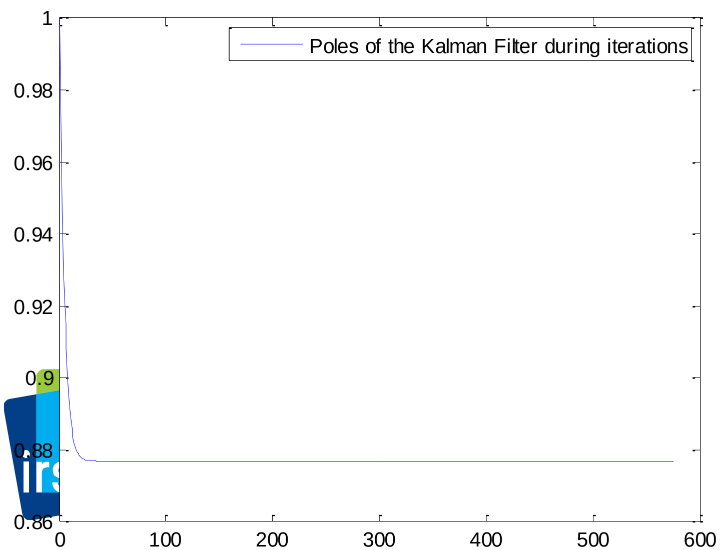
this term is Schur

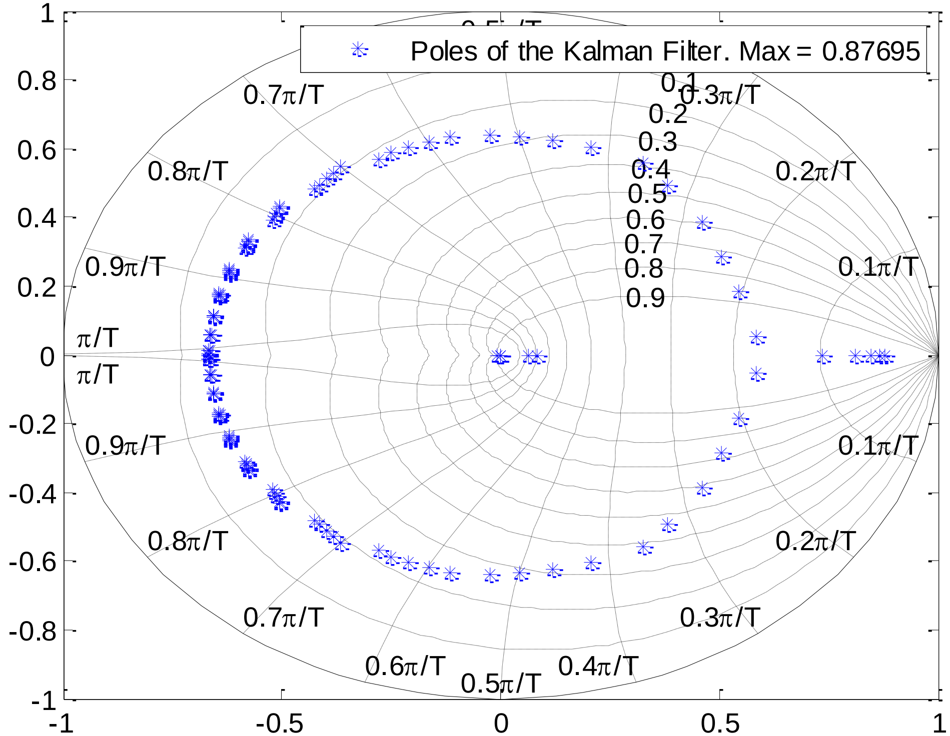
or

=0

With the 10 measurements: Convergence of the reconstruction error -> 0

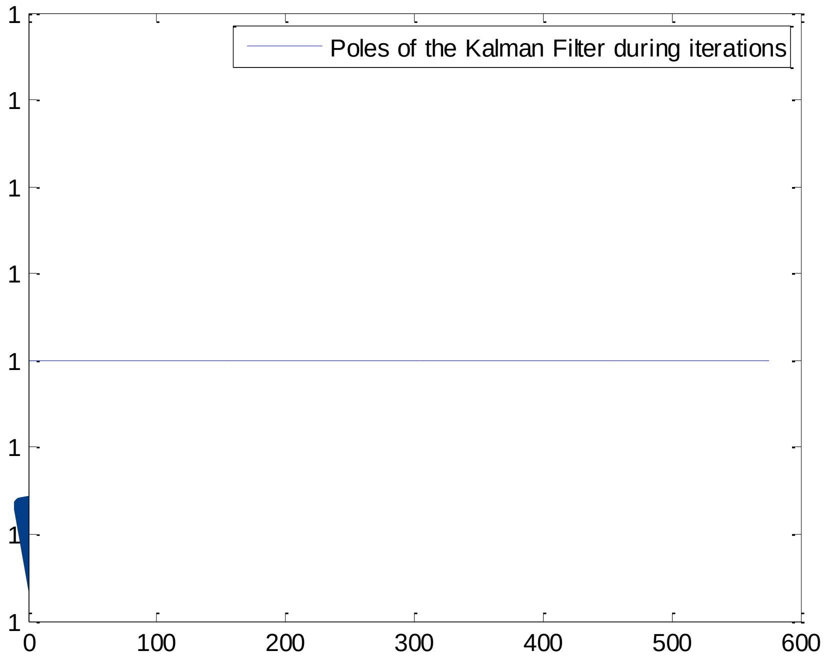
Max eig. (A-KCA) = 0.87695 < 1
Then the error between the estimated state X_a and the true state X_t is converging towards 0 (within the linear assumptions)

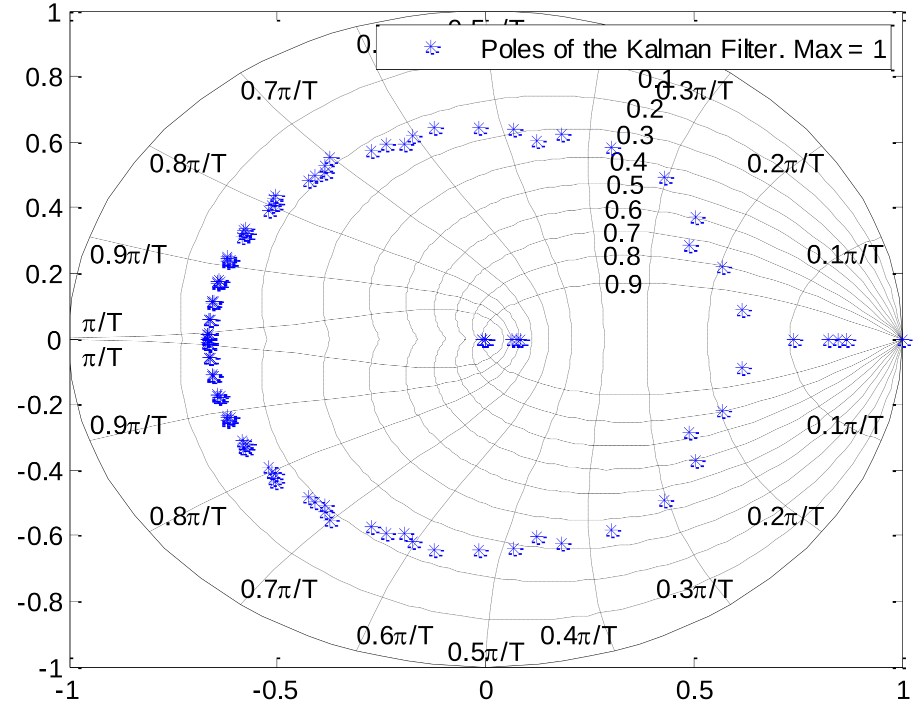




Non convergence if only 9 measurements or less (e.g.: remove y_8)

Max eigenvalue = 1



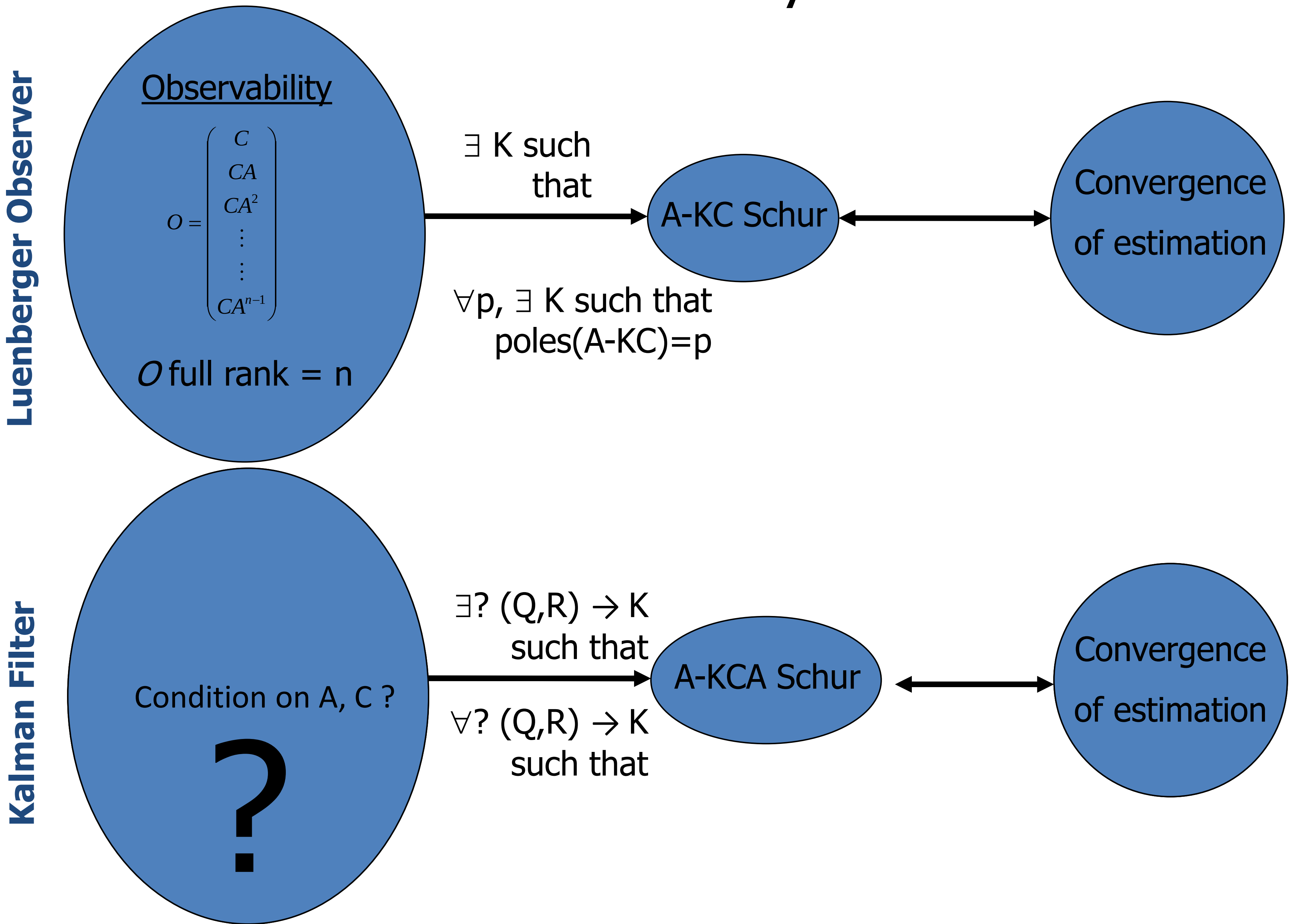


Questions to solve

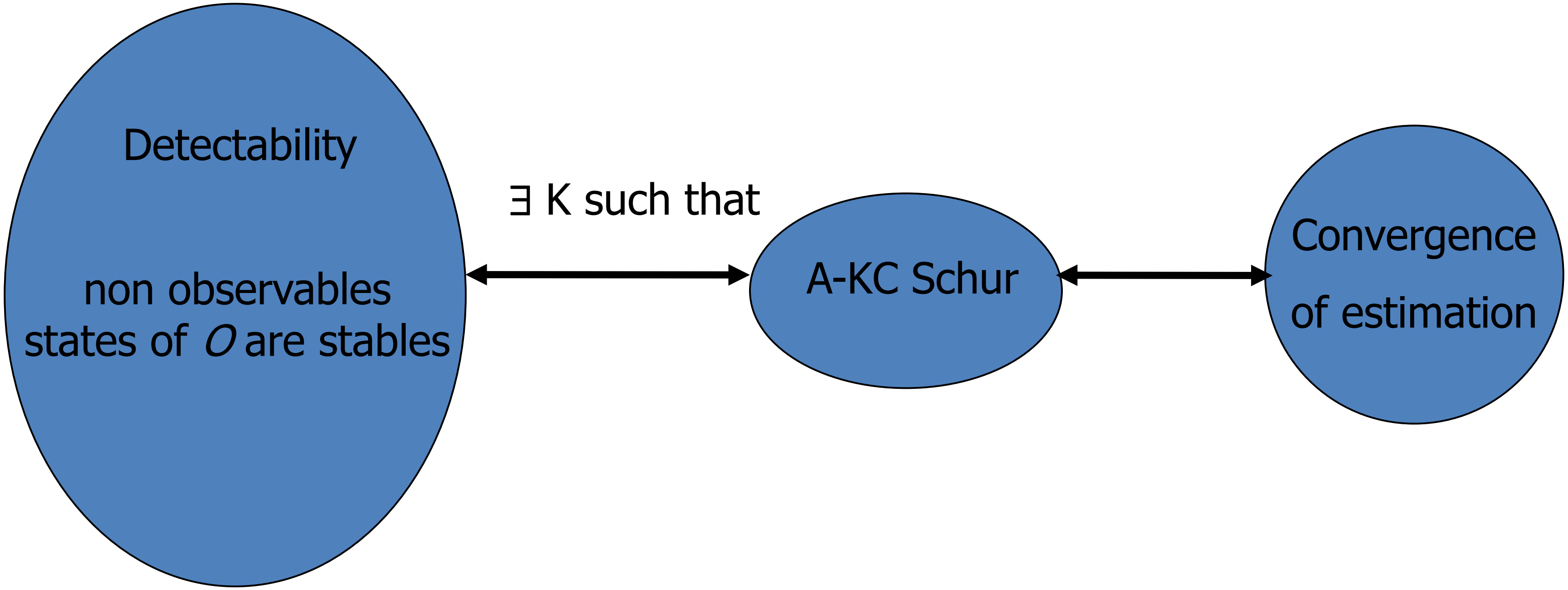
Does it depend only on A, C ?

Is-it possible to choose or analyze (Q,R) so that the resulting gain matrix K, leads to a Schur (A-KCA) matrix ?

Observability



Detectability



Rmk: Same property for A-KCA

Convergence

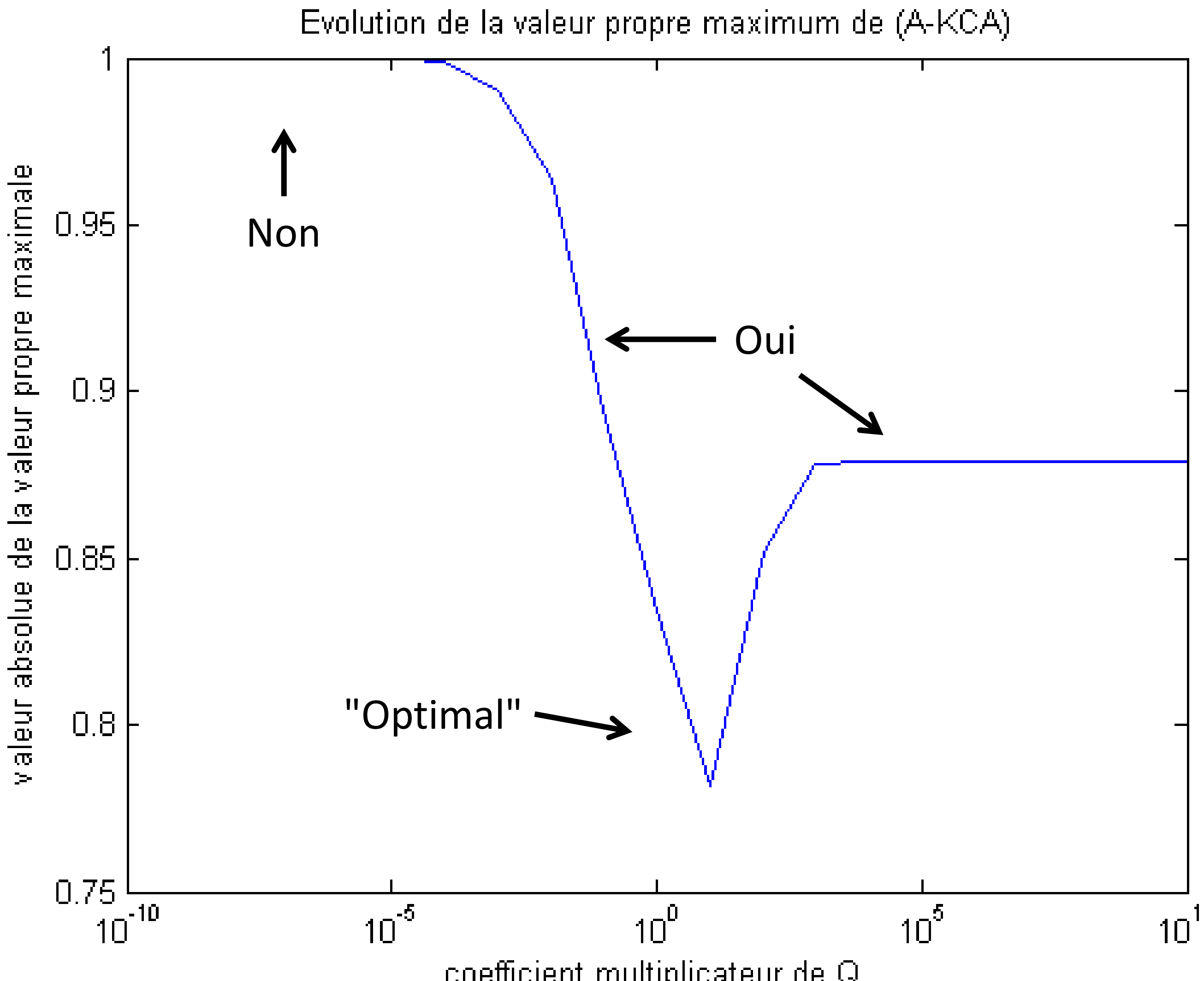
Sufficient condition for convergence :

- Let (A, C) pair be detectable
- Let Q matrix factorizable in $Q=\Gamma\Gamma^T$, such that pair (A, Γ) be stabilizable
- Let R matrix be definite positive

Then the Riccati algebraic equation for the Kalman Gain has 1 and only 1 positive solution and the corresponding matrix (A-KC) is Schur

Rmk: depends on A, C and Q
Def: (A,B) controllable if rank(B AB A^2B …$A^{n-1}B$) = n
Def: stabilizable if non controllable poles are stable

Influence de Q sur l'observabilité



Conclusions 1 (cadre linéaire – FK)

1. Les 10 mesures de niveau permettent d'identifier les 5 Q_p et 5 C_d
2. Le problème inverse est bien posé, et l'erreur de reconstruction de l'état analysé tend bien vers 0
3. Dès qu'on enlève 1 mesure on perd ces résultats : l'erreur ne tends pas vers 0, certaines composantes du vecteur de contrôle ne sont pas bien reconstituées
4. Un test simple a priori (eig value) ou a posteriori (variance) permet de vérifier cette propriété qu'on souhaite
5. L'observabilité (A,C) serait suffisante, mais n'est pas nécessaire. Et difficile à vérifier en grande dimension (nx ≈ 10^2, 10^3)
6. La détectabilité (A,C) + qq autres propriétés sont suffisantes
7. Les covariances des erreurs de modèle Q sont importantes dans ces conditions
8. Le cadre linéaire précédent propose des outils puissants mais a des limites (linéarité, dimension, sensibilité au bruit, etc.) -> 4D-Var pour la suite

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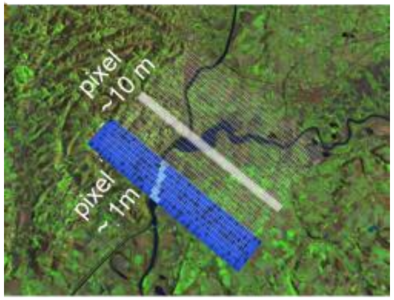
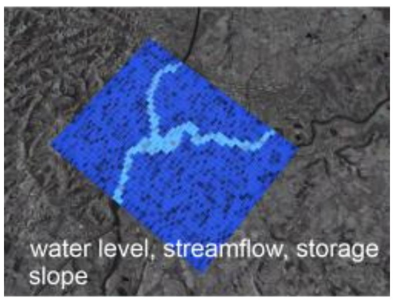
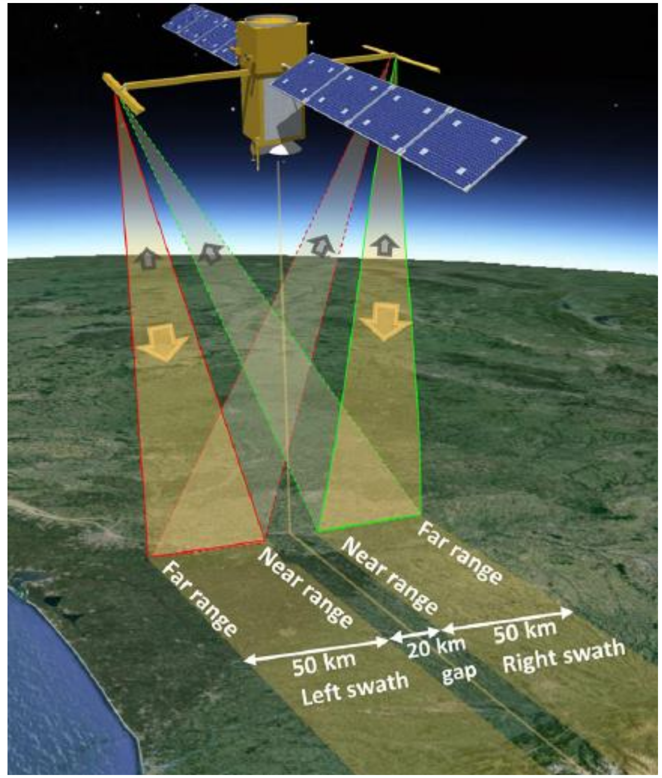
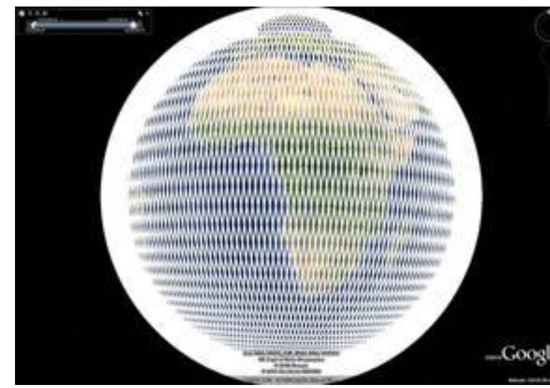
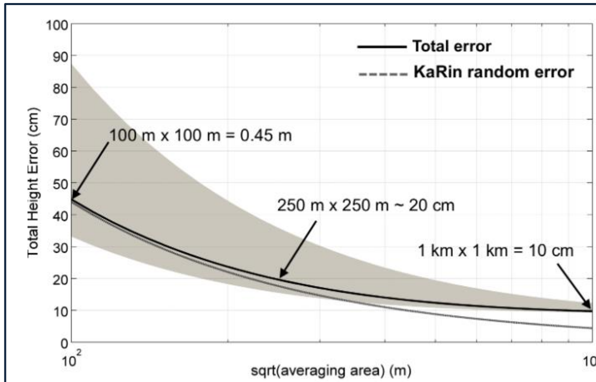
Surface Water and Ocean Topography (SWOT) mission











Scientific requirements	
Observable river width	> 100 m
Height accuracy	10 cm over area > 1 km²
Slope accuracy	1.7 cm/km over area > 1 km
Width accuracy	15% of the evaluated river
Data collection	90% of all ocean/continents within the orbit during 90% of the operational time.

http://www.aviso.altimetry.fr/en

Data assimilation method – Variational approach

- **Observations:** $Y = Y^t + \varepsilon_o \in \mathcal{Y}, \quad \varepsilon_o \sim N(0, \mathcal{O}), \quad \mathcal{R} = E(\varepsilon_o \varepsilon_o^T)$
- **Background:** $U_b = U^t + \varepsilon_b \in \mathcal{U}, \quad \varepsilon_b \sim N(0, \mathcal{B}), \quad \mathcal{B} = E\left(\varepsilon_b \varepsilon_b^T\right)$
- **Control vector:** $U \in \mathcal{U}$

Classical cost function:

$$J(U) = \frac{1}{2}\left\|\mathcal{R}^{-\frac{1}{2}}(\mathcal{G}(U) - Y)\right\|^2 + \frac{1}{2}\left\|\mathcal{B}^{-\frac{1}{2}}(U - U_b)\right\|^2$$

Tikhonov cost function:

$$J(U,\alpha) = \frac{1}{2}\left\|\mathcal{R}^{-\frac{1}{2}}(\mathcal{G}(U) - Y)\right\|^2 + \frac{\alpha}{2}\left\|\mathcal{B}^{-\frac{1}{2}}(U - U_b)\right\|^2,$$

$\alpha > 0$, Tikhonov regularization parameter

Iterative regularization:

$$J(W) = \frac{1}{2}\left\|\mathcal{R}^{-\frac{1}{2}}\left(\mathcal{G}\left(U_b + \mathcal{B}^{-\frac{1}{2}}W, U^0\right) - Y\right)\right\|^2, J(\widetilde{W}) \sim \chi^2(M)$$

$U = U_b + \mathcal{B}^{\frac{1}{2}}W, \qquad M$: Observation space dimension

- $U \in \mathcal{U}$: Control vector
- $U_b \in \mathcal{U}$: Background
- $Y \in \mathcal{Y}$: Observation vector
- $\mathcal{G} : \mathcal{U} \mapsto \mathcal{Y}$: Nonlinear mapping operator
- $\mathcal{B}$: Background covariance matrix
- $\mathcal{R}$: Observation covariance matrix

Data assimilation method – Variational approach

- Limited-memory Broyden–Fletcher–Goldfarb–Shanno (L-BFGS) method :

$$W_{i+1} = W_i + \beta_i \widetilde{H}_i^{-1} \frac{\partial J(W_i)}{\partial W}, W_0 = 0$$

Gradient of the cost function ⇒ Adjoint Model

$$\frac{\partial J(W_i)}{\partial W} = \left(\mathcal{B}^{\frac{1}{2}}\right)^* \left(\mathcal{G}'\left(U_b + \mathcal{B}^{-\frac{1}{2}} W, U^0\right)\right)^* \mathcal{R}^{-1} \left(\mathcal{G}\left(U_b + \mathcal{B}^{-\frac{1}{2}} W, U^0\right) - Y\right)$$

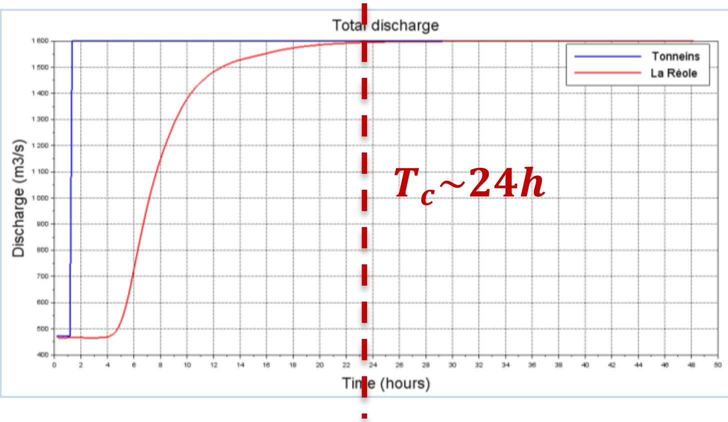
Automatic differentiation TAPENADE (INRIA)

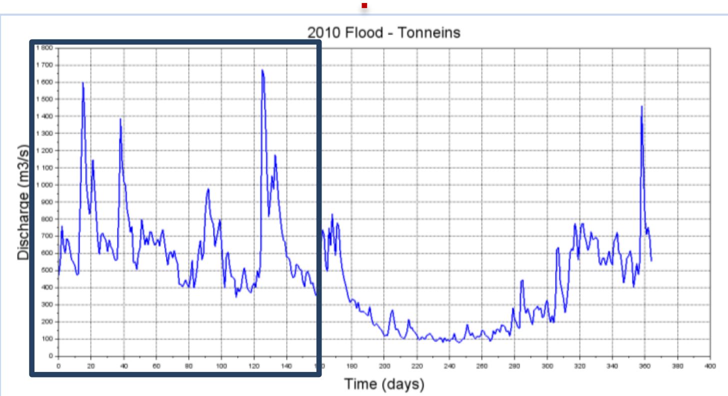
(Gejadze & Malaterre 2016)

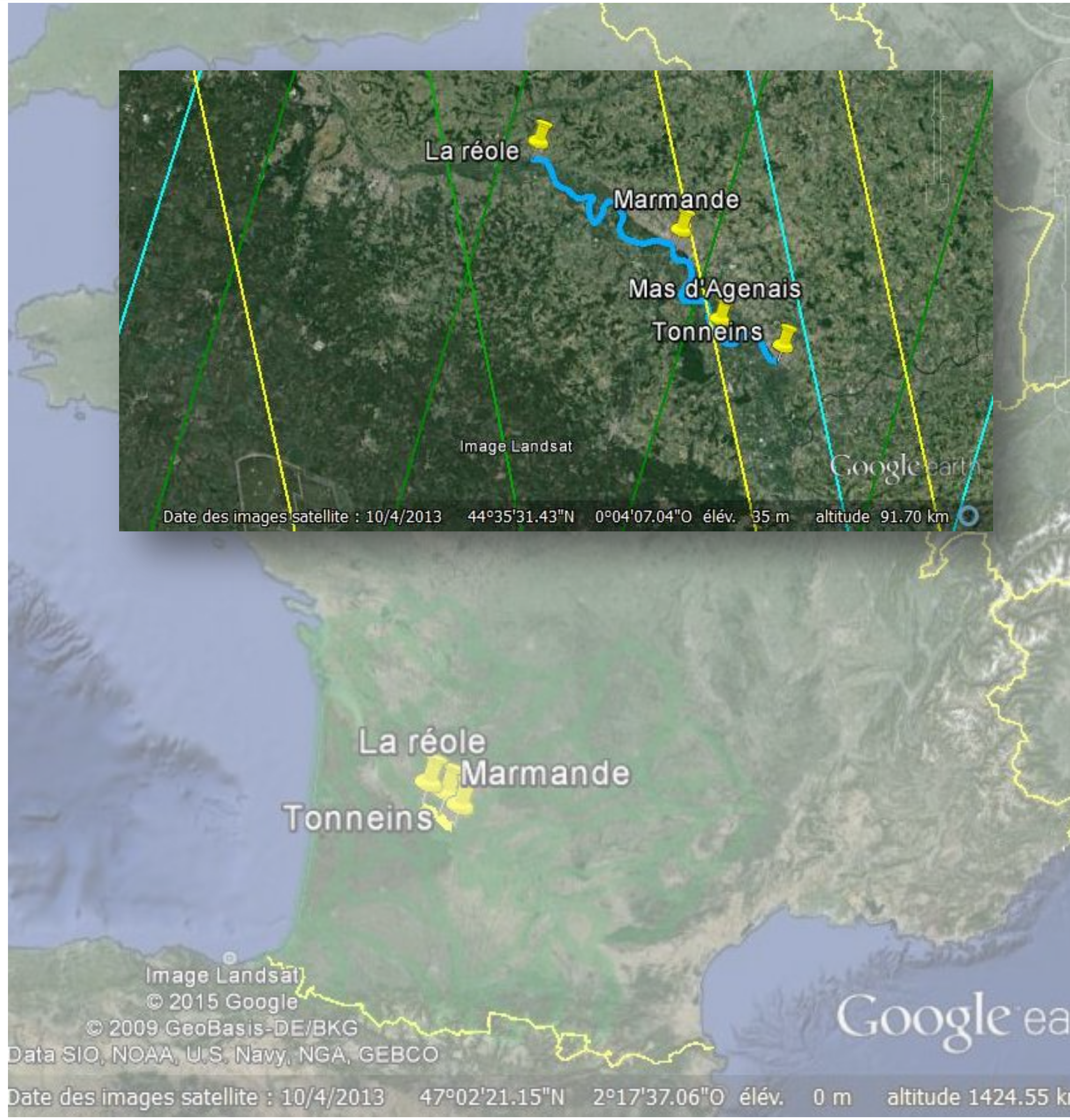
Experimental framework – Study area / period

Garonne River – France

- Downstream reach : 50 km
- Mean width : 170m
- Mean slope : 28cm/km







Experiment (1) : Estimation of Q, given K_S and Z_b

Estimation of upstream discharge *Q* assuming known the bed level Z_b and the friction coefficient K_S, investigating the influence of the SWOT temporal frequency.

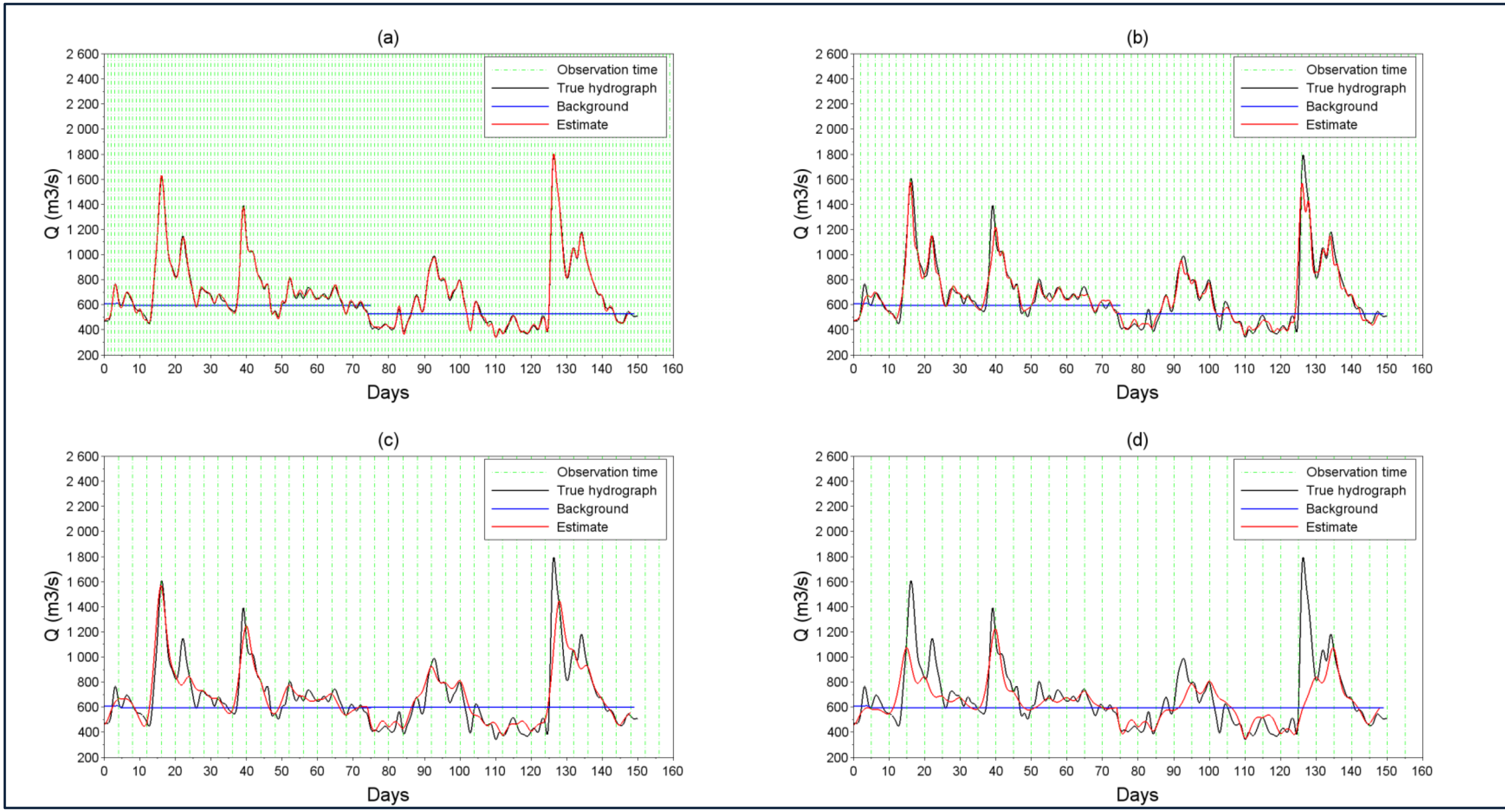
- Identical twin experiments framework
- Assimilation of **water surface elevation *Z*** observations

  Observations error $\boldsymbol{\sigma = 10\ cm}$

  Observations time period: from **1 day to 5 days**

  Observation spatial sampling: each **10 km**
- The first guess on discharge is taken as the **mean annual value**
- Bathymetry and friction assumed **known**
- Sequential version (DA sub-window: 75-day period)
- rRMSE(Q) = $\left(\frac{1}{T}\int_0^T \left(\frac{Q_{estimate}(t)}{Q_{true}(t)} - 1\right)^2 dt\right)^{1/2}$

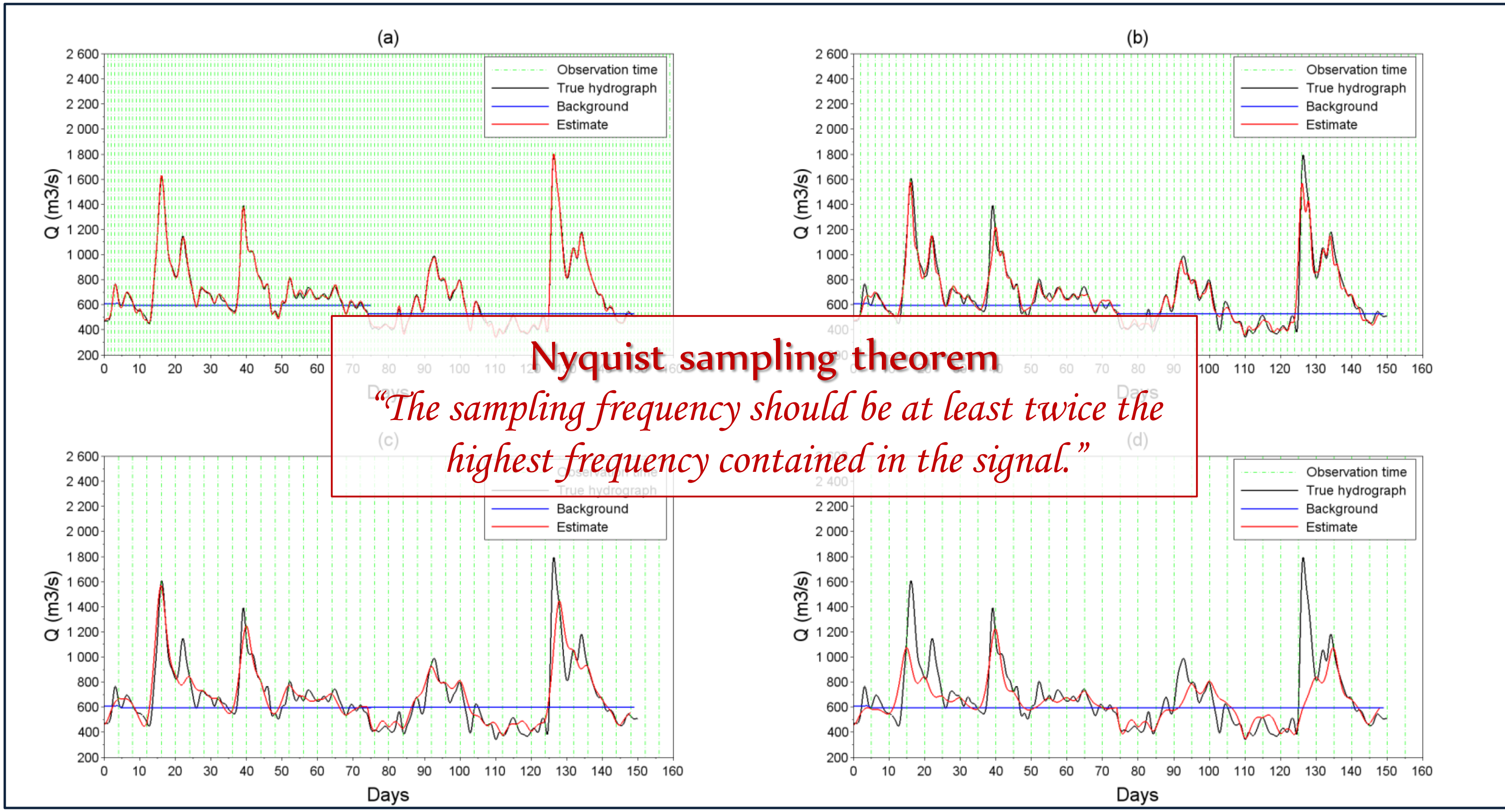
Experiment (1) : Estimation of Q, given K_S and Z_b



Discharge hydrograph at Tonneins from 01/01/2010 to 31/05/2010
(a) 1-day, (b) 2-day, (c) 4-day, (d) 5-day obervation period

	(a)	(b)	(c)	(d)
Q_{rRMSE}	2.1%	9.5%	12.9%	18.2%

Experiment (1) : Estimation of Q, given K_S and Z_b



Discharge hydrograph at Tonneins from 01/01/2010 to 31/05/2010
(a) 1-day, (b) 2-day, (c) 4-day, (d) 5-day obervation period

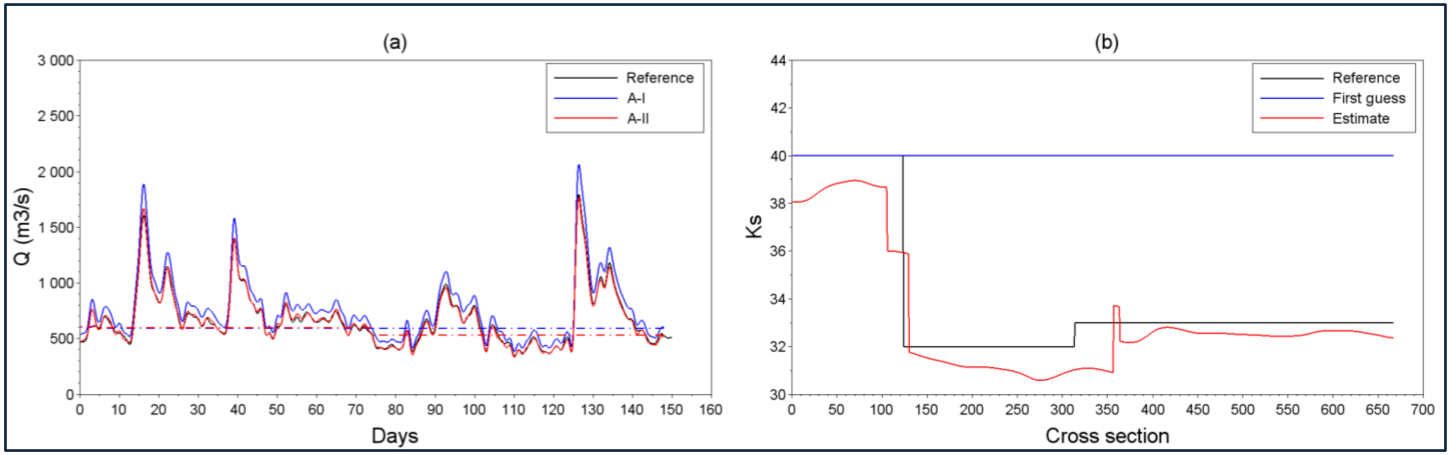
	(a)	(b)	(c)	(d)
Q_{rRMSE}	2.1%	9.5%	12.9%	18.2%

Experiment (2) : Simultaneous estimation of Q and K_S, given exact Z_b

Estimation of upstream discharge *Q* under uncertainty in the friction coefficient *K_S*, assuming known the bed level Z_b

- First guess on the friction coefficient is taken as **a 20% error of the mean value**
- Bathymetry assumed **known**
- Sequential version (DA sub-window: 75-day period)

Experiment (2) : Simultaneous estimation of Q and K_S, given exact Z_b

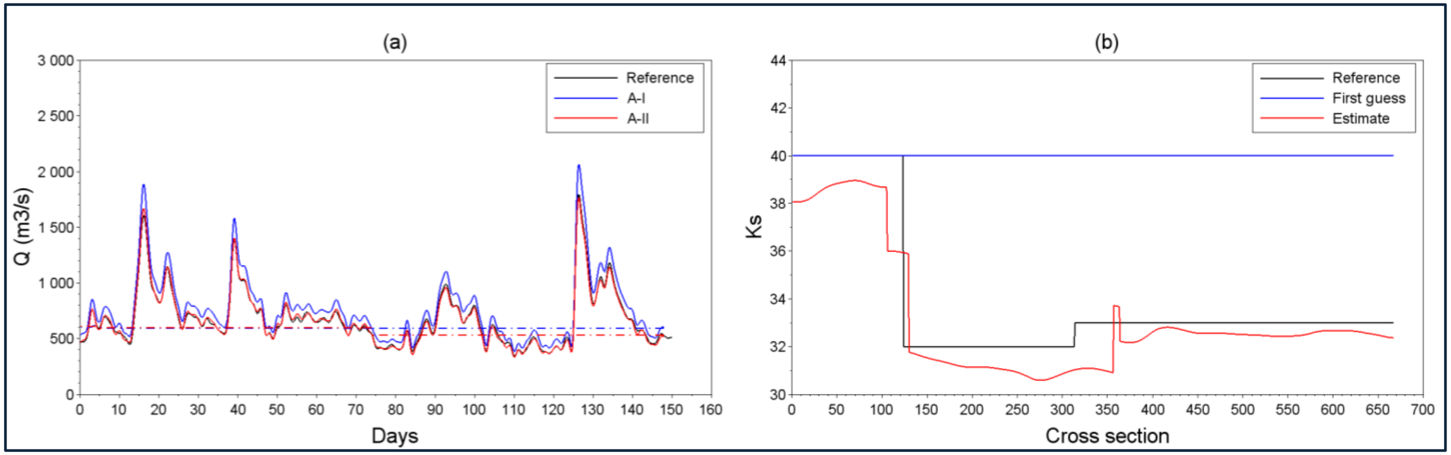


	A-I	A-II
Q_{rRMSE}	12.9%	2.6%
$K_{S_{rRMSE}}$	20.4%	3.4%

A-I : Estimation of Q solely using the first guess on K_S.

A-II : Estimation of Q and K_S.

Experiment (2) : Simultaneous estimation of Q and K_S, given exact Z_b



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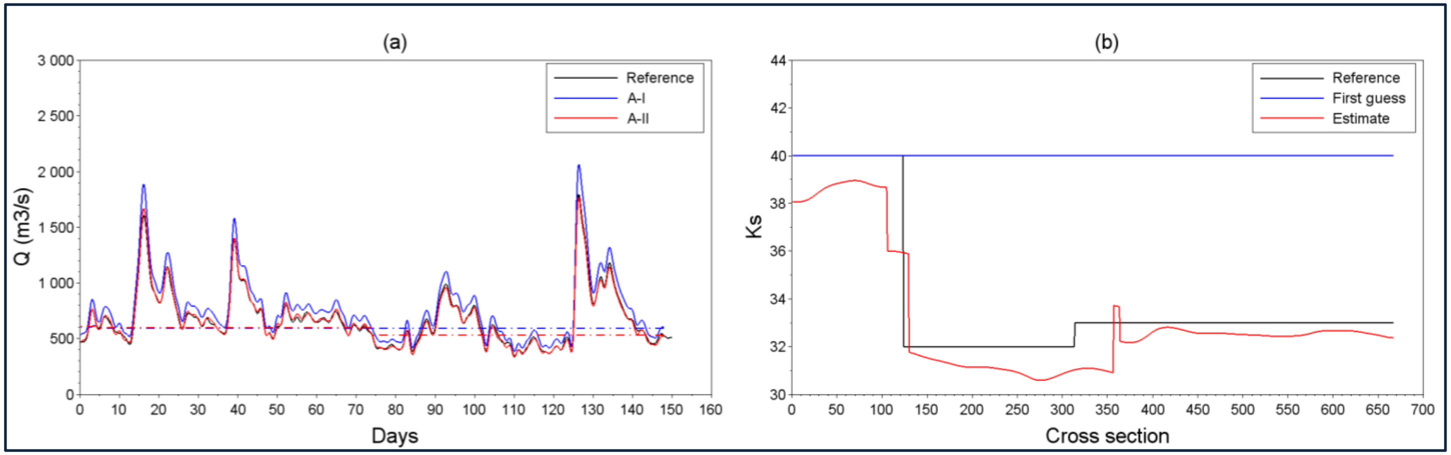
Experiment (3) : Simultaneous estimation of Q and Z_b, given exact K_S

Estimation of upstream discharge Q under uncertainty in the bed level Z_b, assuming known the friction coefficient K_S

- First guess on Bed level is derived from the **perturbed steady flow WSE**
- Friction coefficient and cross-sections shape assumed **known**
- Sequential version (DA sub-window: 75-day period)

Experiment (2) : Simultaneous estimation of Q and K_S, given exact Z_b

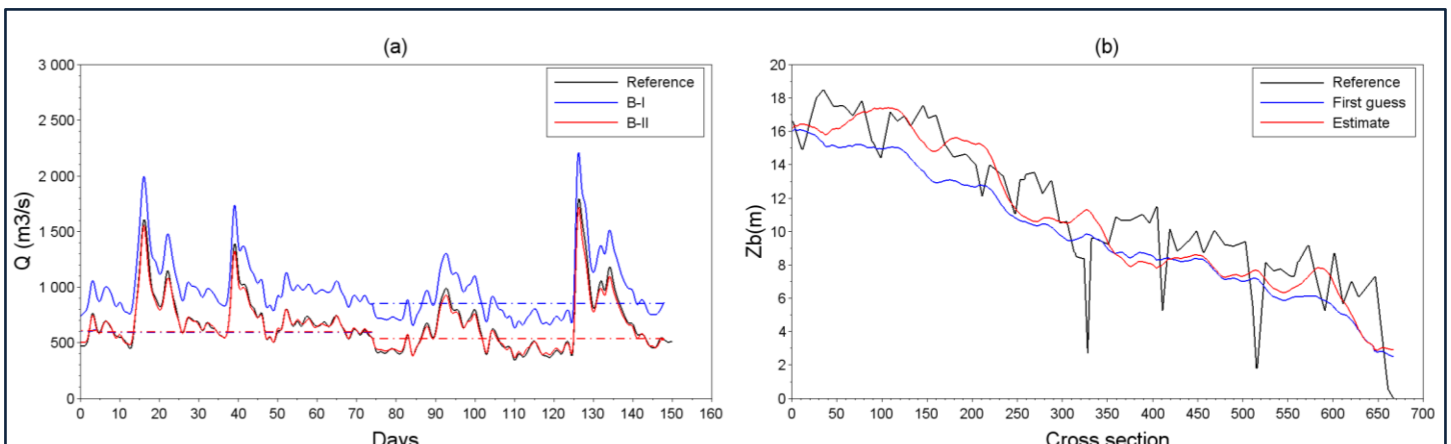


	A-I	A-II
Q_{rRMSE}	12.9%	2.6%
$K_{S_{rRMSE}}$	20.4%	3.4%

A-I : Estimation of Q solely using the first guess on K_S.

A-II : Estimation of Q and K_S.

Experiment (3) : Simultaneous estimation of Q and Z_b, given exact K_S



	B-I	B-II
Q_{rRMSE}	50%	3.8%
$Z_{b_{rRMSE}}$	5.7%	4.9%

B-I : Estimation of Q solely using the first guess on Z_b.

B-II : Estimation of Q and Z_b.

Experiment (4) : Simultaneous estimation of Q, K_S and Z_b

Estimation of upstream discharge Q, under **uncertainty in the bed level Z_b and the friction coefficient K_S**

- Identical twin experiments framework
- Assimilation of **water surface elevation Z** observations

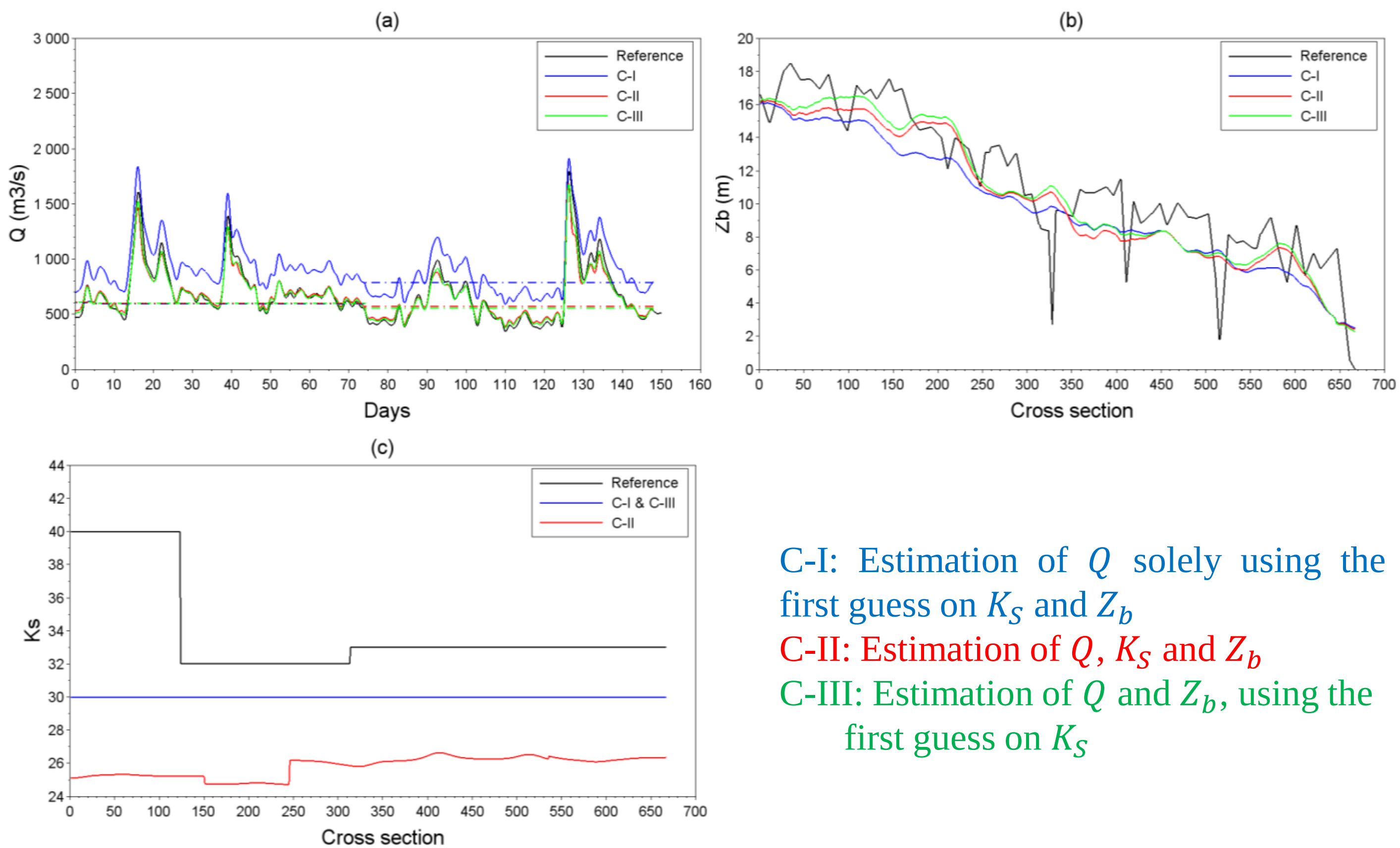
  Observations error $\boldsymbol{\sigma = 10\ cm}$

  Observations time period: **1 day (up to 5 days)**

  Observation spatial sampling: each **10 km**
- The first guess on discharge is taken as the **mean annual value**
- First guess on the friction coefficient is taken as **a 20% error of the mean value**
- First guess on Bed level is derived from the **perturbed steady flow WSE**
- Cross-sections shape assumed **known**

- Sequential version (DA sub-window: 75-day period)

Experiment (4) : Simultaneous estimation of Q, K_S and Z_b



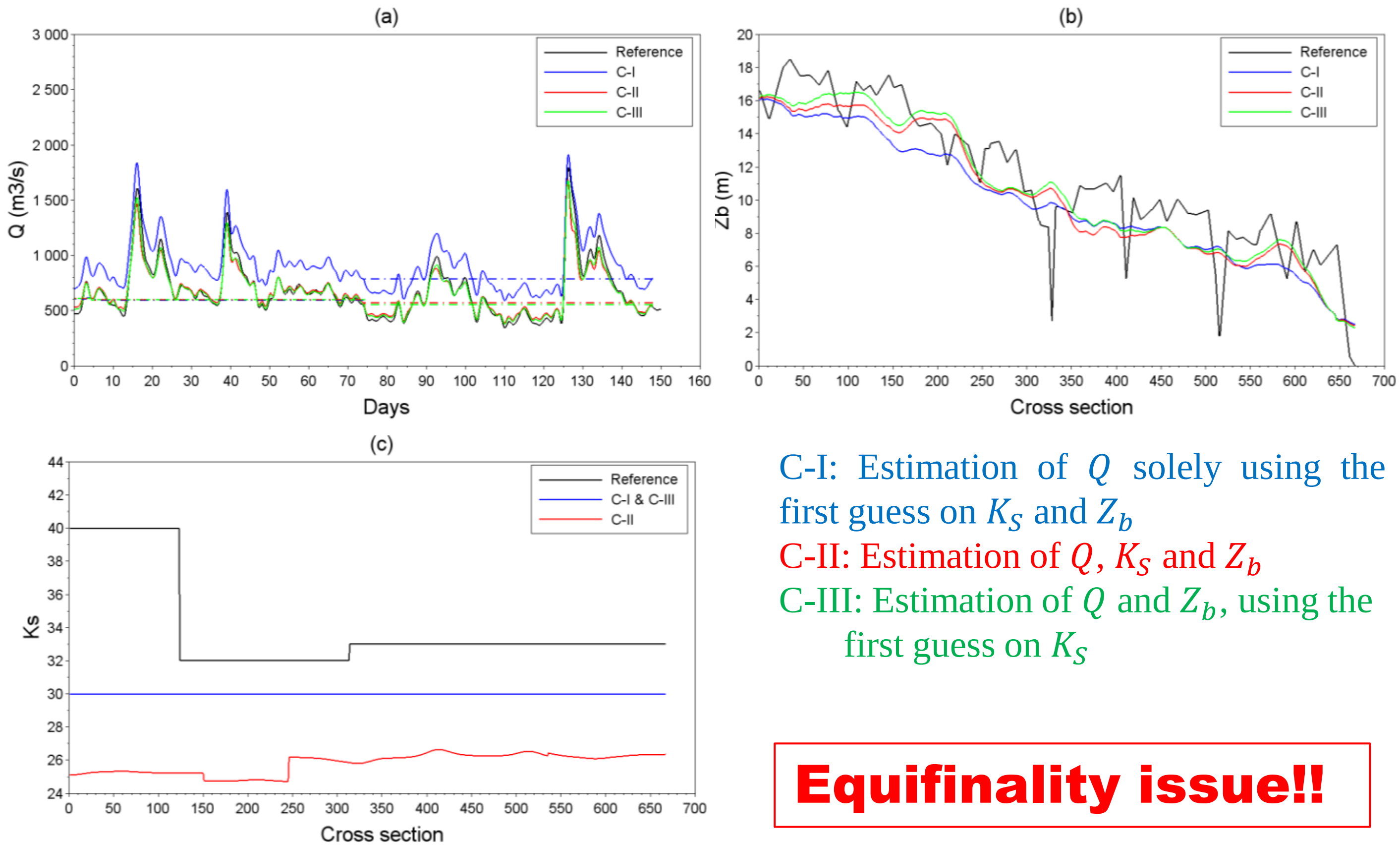
C-I: Estimation of Q solely using the first guess on K_S and Z_b

C-II: Estimation of Q, K_S and Z_b

C-III: Estimation of Q and Z_b, using the first guess on K_S

rRMSE	**C-I**	**C-II**	**C-III**
Q	40.5%	7.1%	5.1%
K_S	13%	24.4%	13%
Z_b	5.7%	4.7%	4.5%

Experiment (4) : Simultaneous estimation of Q, K_S and Z_b



C-I: Estimation of Q solely using the first guess on K_S and Z_b

C-II: Estimation of Q, K_S and Z_b

C-III: Estimation of Q and Z_b, using the first guess on K_S

Equifinality issue!!

$rRMSE$	C-I	C-II	C-III
Q	40.5%	7.1%	5.1
K_S	13%	24.4%	13%
Z_b	5.7%	4.7%	4.5%

Control set design: notations

$U \in \mathcal{U}$ - **full set** of model inputs (including auxiliary)

$X \in \mathcal{X}$ - state variables

$$\underline{X = \mathcal{M}(U)} \text{ - model or input-to-state mapping } \mathcal{M} : \mathcal{U} \to \mathcal{X}$$

$\bar{U}$ - 'true'/exact input , $\bar{X} = \mathcal{M}(\bar{U})$ - 'true' model prediction

$U^* = \bar{U} + \eta$, where U^* - background/prior , η - background error

$G \in \mathcal{G}$ - goal-function or Quantity of Interest (QoI)

$$\underline{G = D(X)} \text{ - state-to-design mapping } D : \mathcal{X} \to \mathcal{G}$$

$$\boxed{\delta G} = D(X) - D(\bar{X}) = \boxed{D(\mathcal{M}(U)) - D(\mathcal{M}(\bar{U}))} \text{ - goal-function error}$$

$Y \in \mathcal{Y}$ - observables

$$\underline{Y = C(X) = C(\mathcal{M}(U)) := R(U)}$$

- input-to-observations mapping $R : \mathcal{U} \to \mathcal{Y}$

$\bar{Y} = R(\bar{U})$ - 'true'/exact observations

$Y^* = \bar{Y} + \xi = R(\bar{U}) + \xi$, ξ - observation error

$\hat{U} = U | Y^*$ - estimate of U, posterior

Control set design: variational DA

Variational data assimilation:

$$J(U) = \frac{1}{2}\|O^{-1/2}(R(U) - Y^*)\|^2_{\mathcal{Y}} + \frac{1}{2}\|B^{-1/2}(U - U^*)\|^2_{\mathcal{U}} \to \inf_U$$

$O = E[\xi\xi^T]$ - observation error covariance

$B = E[\eta\eta^T]$ - background error covariance

$J'_U(\hat{U}) = 0$ - optimality condition

$(R'_U(\hat{U}))^* O^{-1}(R(\hat{U}) - Y^*) + B^{-1}(\hat{U} - U^*) = 0$ - estimator equation

where $R'(\cdot)$ - TLM, $(R'(\cdot))^*$ - adjoint

$(R'_U(\hat{U}))^* O^{-1}(R'_U(\tilde{U})\delta U - \xi) + B^{-1}(\delta U - \eta) = 0$ - error equation

δU, $V_{\delta U}$ - estimation error covariance

$\delta G = D(\mathcal{M}(\hat{U})) - D(\mathcal{M}(\bar{U}))$ - posterior goal-function error

$V_{\delta G} := E[\delta G \delta G^T] = D'_X(\bar{X})\mathcal{M}'_U(\bar{U})V_{\delta U}(\mathcal{M}'_U(\bar{U}))^*(D'_X(\bar{X}))^*$

- goal-function error covariance for small δU

Control set design: goal-function covariance

$$V_{\delta G} := E[\delta G \delta G^T] = D'_X(\bar{X})\mathcal{M}'_U(\bar{U}) V_{\delta U} (\mathcal{M}'_U(\bar{U}))^* (D'_X(\bar{X}))^*$$

$$V_{\delta U} = B \text{ - without DA}$$

$$V_{\delta U} \simeq H^{-1}(\bar{U}) \simeq H^{-1}(\hat{U}) \text{ - after DA}$$

$$H(U) = (R'_U(\cdot))^* O^{-1} R'_U(\cdot) + B^{-1} \text{ - Hessian of an auxiliary control problem}$$

It is impossible to control the full input U !

1. Technical/implementation issues:
   a) dimension of the control vector
   b) convergence rate
2. Fundamental:
   a) identifiability / equifinality;
   b) convexity;
   c) connectivity of the solution domain

Aim: design of a sufficient control set

tradeoff: accuracy versus robustness and solvability

Control set design: partial control case

$U_a \in A$ - active subset of the full control vector U

$U_p = U \setminus U_a$ - passive subset of U

Assumption: B is block-diagonal with block B_a and B_p

Variational data assimilation:

$$J(U_a) = \frac{1}{2}\|O^{-1/2}(R(U_a, U_p^*) - Y^*)\|^2_{\mathcal{Y}} + \frac{1}{2}\|B_a^{-1/2}(U_a - U_a^*)\|^2_A \to \inf_{U_a}$$

$J'_{U_a}(\hat{U}_a) = 0$ - optimality condition for U_a

$$(R'_{U_a}(\hat{U}_a, U_p^*))^* O^{-1}(R(\hat{U}_a, U_p^*) - Y^*) + B^{-1}(\hat{U}_a - U_a^*) = 0$$

- estimator equation

$$(R'_{U_a}(\hat{U}_a, U_p^*))^* O^{-1}(R'_{U_a}(\tilde{U}_a, U_p^*)\delta U_a + R'_{U_p}(\bar{U}_a, \tilde{U}_p^*)\eta_p - \xi) + B_a^{-1}(\delta U_a - \eta_a) = 0$$

- error equation

δU_a - estimation error

$\delta U = (\delta U_a, \eta_p)^T$ - posterior input vector

$V_{\delta U} = E[\delta U \delta U^T]$ - estimation error covariance

Control set design: partial control covariance

$$V_{\delta G} := E[\delta G \delta G^T] = D'_X(\bar{X})\mathcal{M}'_U(\bar{U})V_{\delta U}(\mathcal{M}'_U(\bar{U}))^*(D'_X(\bar{X}))^*$$

- goal-function error covariance for small δU

Covariance of δU:

$$V_{\delta U} = \begin{pmatrix} V_{\delta U_a} & V_{\delta U_{ap}} \\ V_{\delta U_{pa}} & B_p \end{pmatrix}$$

$$V_{\delta U_a} = E[\delta U_a \delta U_a^T] = H_a^{-1} + H_a^{-1}(R'_{U_a})^* O^{-1} R'_{U_p} B_p (R'_{U_p})^* O^{-1} R'_{U_a} H_a^{-1}$$

$$V_{\delta U_{ap}} = E[\delta U_a \eta_p^T] = -H_a^{-1}(R'_{U_a})^* O^{-1} R'_{U_p} B_p$$

$$V_{\delta U_{pa}} = E[\eta_p \delta U_a^T] = -B_p (R'_{U_p})^* O^{-1} R'_{U_a} H_a^{-1}$$

where $\quad H_a(\bar{U}) = (R'_{U_a}(\bar{U}))^* O^{-1} R'_{U_a}(\bar{U}) + B_a^{-1}$

- Hessian of an auxiliary control problem for U_a

Control set design: implementation

$H(U) = (R'_U(\cdot))^* O^{-1} R'_U(\cdot) + B^{-1}$ - Hessian of an auxiliary control problem

$$\tilde{H} = (B^{1/2})^* H B^{1/2} = \begin{pmatrix} \tilde{H}_a & \tilde{H}_{ap} \\ \tilde{H}_{pa} & \tilde{H}_p \end{pmatrix}$$

All components of $V_{\delta U}$ are defined using $\tilde{H}$ as follows:

$$V_{\delta U_a} = B_a^{1/2} \tilde{H}_a^{-1/2} (I_a + \tilde{H}_a^{-1/2} \tilde{H}_{ap} \tilde{H}_{pa} \tilde{H}_a^{-1/2}) \tilde{H}_a^{-1/2} (B_a^{1/2})^*$$

$$V_{\delta U_{ap}} = -B_a^{1/2} \tilde{H}_a^{-1} \tilde{H}_{ap} (B_p^{1/2})^*$$

$$V_{\delta U_{pa}} = -B_p^{1/2} \tilde{H}_{pa} \tilde{H}_a^{-1} (B_a^{1/2})^*$$

All components of $V_{\delta U}$ are defined using (λ_i, W_i) of

$\tilde{H}^\gamma(\bar{U}) \simeq I + \sum_{i=1}^{L_H} (\lambda_i^\gamma - 1) W_i W_i^*$ - limited-memory representation of $\tilde{H}$

Control set design: algorithm

1. Compute a set (λ_i, W_i) of the largest eigen(values/vectors) of $H(U)$
2. Design loop:
   2.1 Choose active set U_a

   2.2 Compute a set of the largest eigen(values/vectors) of $\tilde{H}_a(U_a)$

   2.3 Compute a set $(\lambda_{G,i}, W_{G,i})$ of the largest eigen(values/vectors) of $V_{\delta G}$

   2.4 Evaluate $V_{\delta G} = \sum_{i=1}^{L_G} \lambda_{G,i} W_{G,i} W_{G,i}^T$
- End of design loop

Remarks:

Eigenvalue decomposition is computed using the Lanczos method

Numerical models (TLM/adjoint) are involved at:

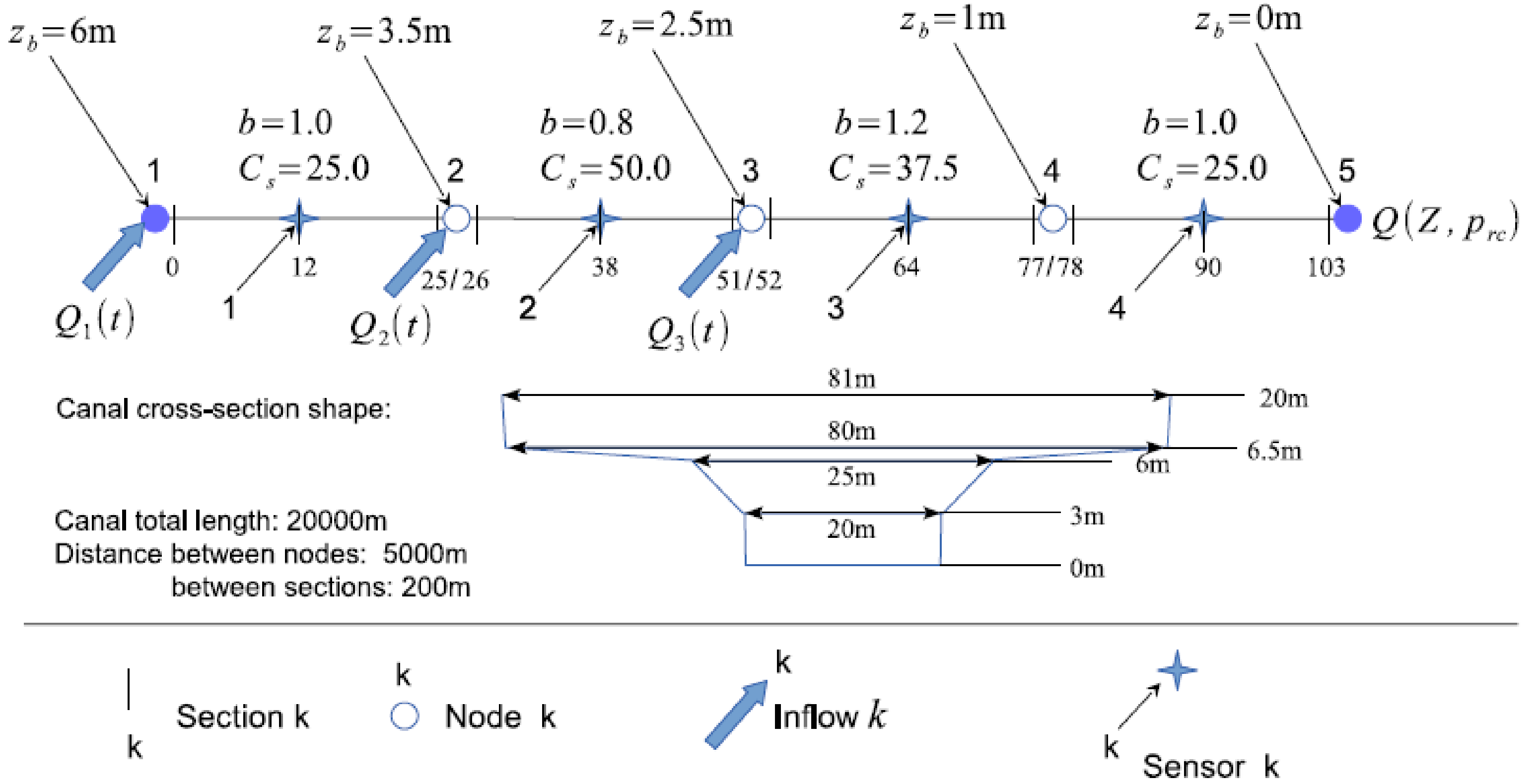
- step 1 (operators $R', (R')^*$)
- step 2.3 (operators $M'D', (D')^*(M')^*$)

Control set design: test configuration

Particular form of QoI's:

$$G_Q(x) = \int_{t_1}^{t_2} |Q(x,t) - Q_*(x)|\, dt \qquad G_Z(x) = \int_{t_1}^{t_2} |Z(x,t) - Z_*(x)|\, dt$$

Standard deviation: $\sigma[\delta G] = diag(V_{\delta G})$



Control set design: NA results, case A

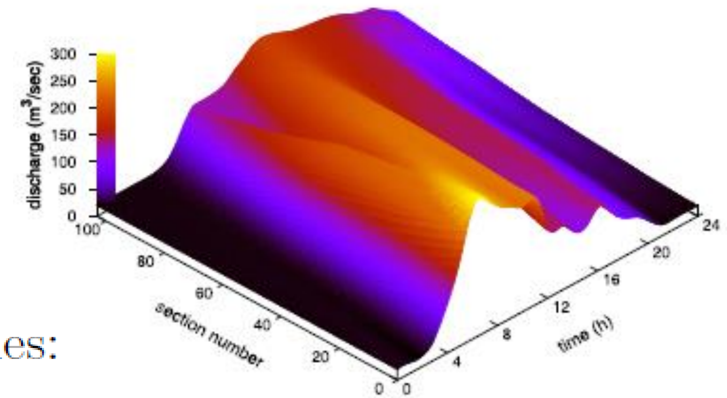
$$U = (Q_1(t), z_b(k), b(k), C_s(k), U_*)^T, \; k = 1, \ldots, K_s$$

$cc1$ – full control case: i.e. $U_a = U \setminus U^*$;

$cc2$ – partial control case: $U_a = Q_1(t)$;

$cc3$ – partial control case: $U_a = (Q_1(t), z_b(k))^T$;

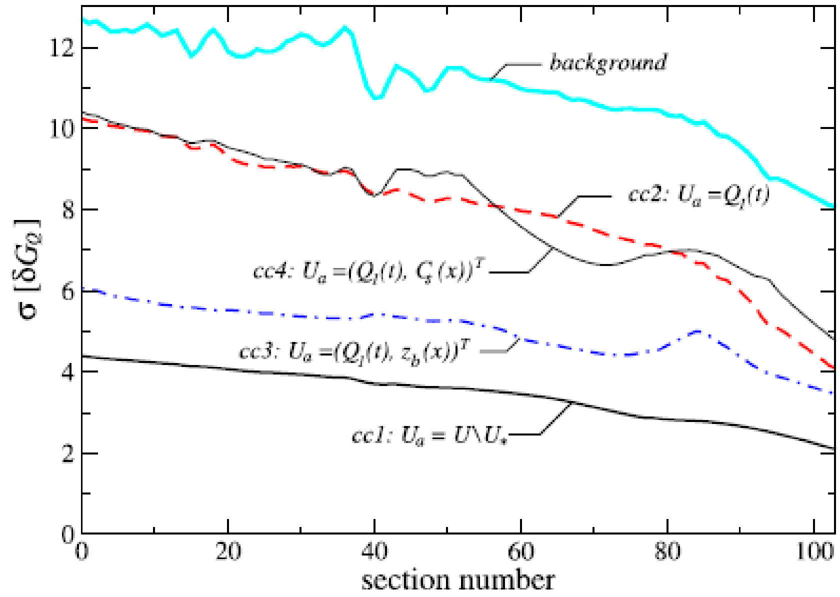
$cc4$ – partial control case: $U_a = (Q_1(t), C_s(k))^T$

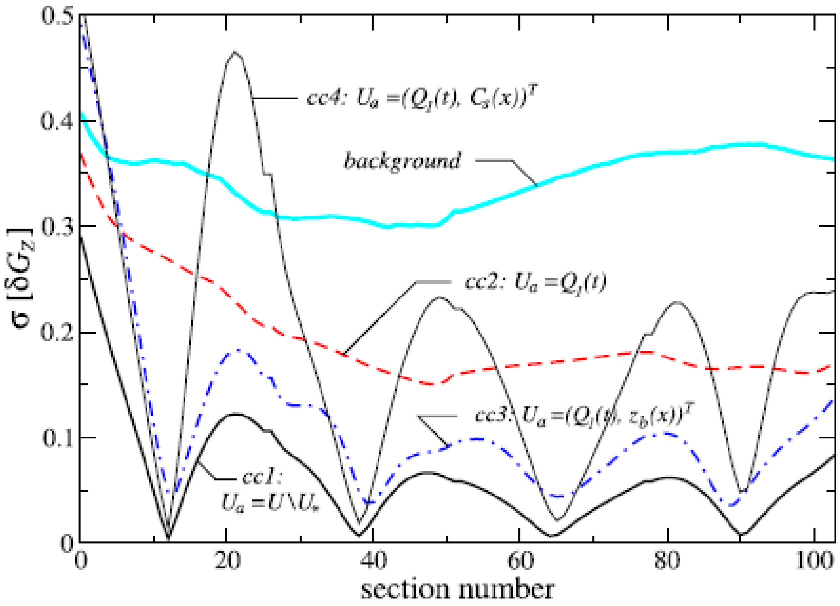


Background/prior st. dev. for uncertainties:

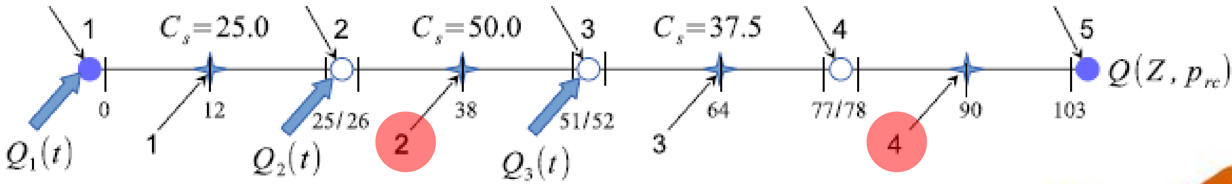
$\sigma[\delta Q_1] = 25m^3/s$

$\sigma[\delta z_b] = 0.33m, \; \sigma[\delta C_s] = 3.3 \;, \; \sigma[\delta b] = 0.07$



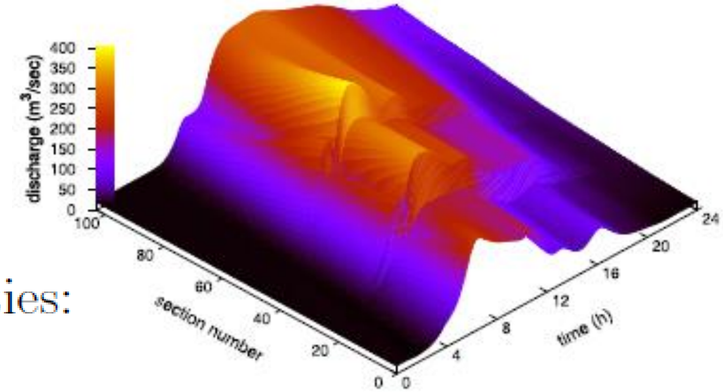


Control set design: NA results, case B



$$U = (Q_1(t), Q_2(t), Q_3(t), U_*)^T$$

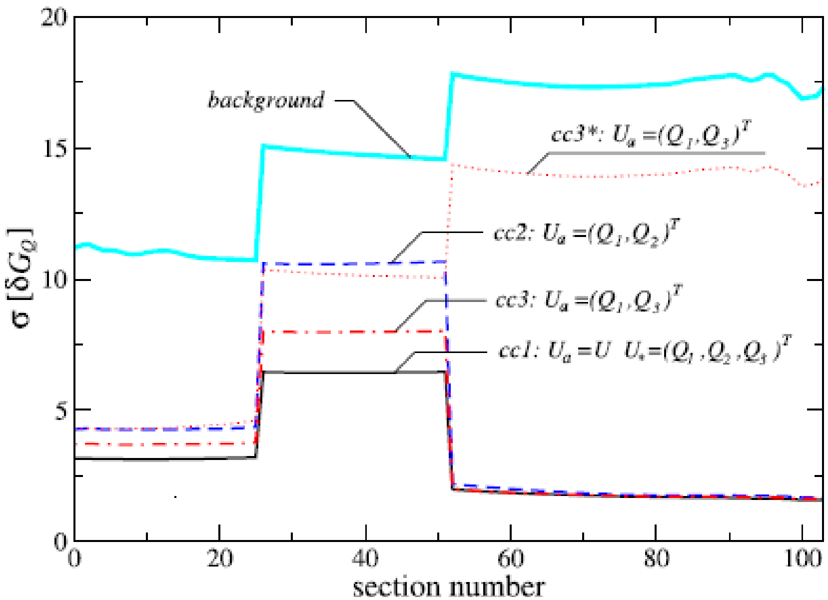
$cc1$ – full control case: i.e. $U_a = U \setminus U^* = (Q_1(t), Q_2(t), Q_3(t))^T$;
$cc2$ – partial control case: $U_a = (Q_1(t), Q_2(t))^T$;
$cc3$ – partial control case: $U_a = (Q_1(t), Q_3(t))^T$;

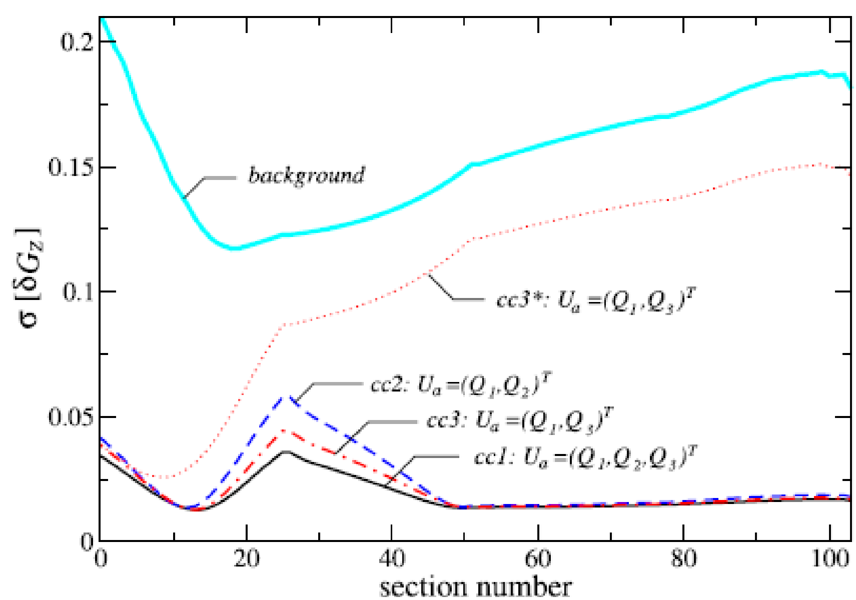


Background/prior st. dev. for uncertainties:

$$\sigma[\delta Q_1, \delta Q_2, \delta Q_3] = 25m^3/s$$

$$\sigma[\delta z_b, \delta C_s, \delta b] = 0$$





Conclusion 2 (cadre variationnel)

1. Le concept de choix du vecteur de contrôle a été présenté. Ce concept est utile pour les modèles ayant des incertitudes nombreuses, complexes parmi leurs entrées. En particulier, les erreurs de modèle peuvent être incluses dans ces entrées du modèle.

2. La méthode présentée permet de quantifier la performance obtenue pour toute combinaison d'entrées actives (sous-partie du vecteur des entrées), permettant de faire apparaitre les sous-vecteurs suffisants . Le choix entre ces options de sous-vecteurs suffisants peut ensuite être fait selon des considérations de solvabilité et de robustesse.

3. La méthode est une généralisation de l'approche variationnelle classique de Quantification ou Réduction des Incertitudes. Elle n'utilise pas de formalisme matriciel et est donc adaptée aux modèles de grande dimension.

4. La méthode a été appliquée dans le domaine de l'hydraulique à surface libre et a prouvé son intérêt. Les résultats obtenus illustrent et justifient toutes les conclusions que nous avions tirées de nos tests d'AD avec SIC (et son adjoint).

5. La même approche peut être étendue au cas ou les entrées (vecteur de contrôle) sont divisées entre une part active, passive et de nuisance. Des développements peuvent aussi être proposés pour obtenir une performance globale plutôt que locale.

Thanks ! Questions ?

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